Grade 6 Math Circles

Week of 20th November

Matrices - Problem set

1. Given the following matrices, determine if the following operations are possible, and if so, state the resulting dimension of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad E = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

   (a) $5A$
   (b) $B + E^T$
   (c) $C + 2E$
   (d) $-3A + D^T$
   (e) $A + 4B$

2. Given the following matrices, compute the following expressions.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 10 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

   (a) $5A - B^T$
   (b) $3B + C$
   (c) $A + (B - 2C)$
   (d) $B - (2A + C)$

3. In this lesson, we learnt the determinant, an operation that acts on matrices. There exists another, called the Trace. The trace is defined as the sum of the diagonals of a matrix. Say we have a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\text{Tr}\,(A) = \text{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

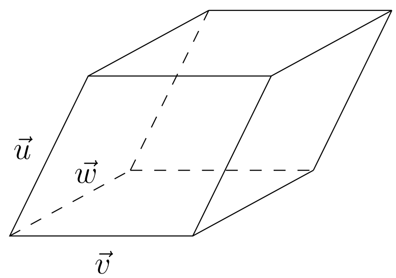
   Determine the Trace of the following matrices

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 3 & 1 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 9 & 4 \\ 1/3 & 7/6 \end{bmatrix}$

4. Like we did with the area of a parallelogram, we can also determine the volume of a parallelepiped. Given the three vectors: $\vec{u}$, $\vec{v}$, and $\vec{w}$ that make up the sides of the parallelepiped,



the volume of this solid is

$$\text{Volume} = \left|\det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\right|$$

The formula for the determinant of a 3×3 matrix is as follows

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Given the following vectors, determine the volume of the corresponding parallelepiped

(a) $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

(b) $\vec{u} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ -1/2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

(c) $\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$

5. Find the area of the parallelogram given by the vectors $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Explain your answer.

6. We can also multiply two matrices. To perform matrix multiplication, we multiply each element of a row from one matrix by the corresponding element of the column from the other matrix and add the product together. For example, multiplying two 2×2 matrices looks like

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Compute the following matrix multiplications

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \end{bmatrix}$

*Do you always get a square matrix when you compute matrix multiplication? What is the resultant dimension of two multiplied matrices?

7. Let $\vec{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find the resulting vector from rotating $\vec{p}$ by 90° and then reflecting across the line $y = x$. Does the order in which you apply the transformation matter?