Math Circles - Intro to Combinatorics - Winter 2024

Problem Set 2

February 14th, 2024

1. Find the expansion of $(x+4y)^3$.
   Solution: $(x+4y)^3 = \sum_{i=0} 3\binom{3}{i}x^{3-i}(4y)^i = \binom{3}{0}x^3(4y)^0 + \binom{3}{1}x^2(4y) + \binom{3}{2}x(4y)^2 + \binom{3}{3}x^0(4y)^3 = x^3 + 3x^2(4y) + 3x(16y) + 64y^3 = x^3 + 12x^2y + 48xy^2 + 64y^3$.
2. Find the expansion of $(x+\frac{1}{x})^5$.
   Solution: $(x+\frac{1}{x})^5 = \sum_{i=0} 5\binom{5}{i}x^{5-i}\frac{1}{x}^i = \binom{5}{0}x^5\frac{1}{x}^0 + \binom{5}{1}x^4\frac{1}{x}^1 + \binom{5}{2}x^3\frac{1}{x}^2 + \binom{5}{3}x^2\frac{1}{x}^3 + \binom{5}{4}x^1\frac{1}{x}^4 + \binom{5}{5}x^0\frac{1}{x}^5 = x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$.
3. Find the coefficient of x^4y^2 in $(x+2y)^6$.
   Solution: Using binomial theorem we have $\binom{6}{4}x^4(2y)^2 = 15x^4(2y)^2 + 60x^4y^2$. SO the coefficient is 60
4. Find the coefficient of x^6 in $(x^2+1)^5$.
   Solution: Notice the x is squared so we want the third power of x^2. Thus we have $\binom{5}{3}(x^2)^3(1)^2 = 10x^6$. So the coefficient is 10.
5. Find the coefficient of x^6y^3 in $(x^3-2y)^5$
   Solution: We want the second power of the x term. So $\binom{5}{2}(x^3)^2(-2y)^3 = 10x^6(-8)y^3 = -80x^6y^3$. Thus the coefficient is -80.
6. Expand $(x-\sqrt{2})^4$.
   Solution: $(x-\sqrt{2})^4 = \sum_{i=0}^{4}\binom{4}{i}x^{4-i}(-\sqrt{2})^i = \binom{4}{0}x^4(-\sqrt{2}) + \binom{4}{1}x^3(-\sqrt{2}) + \binom{4}{2}x^2(-\sqrt{2})^2 + \binom{4}{3}x^1(-\sqrt{2})^3 + \binom{4}{4}x^0(-\sqrt{2})^4 = x^4 - 4\sqrt{2}x^3 + 12x^2 + -8\sqrt{2}x + 4$.
7. Pattern Investigation
   (a) Write down any patterns you have noticed in the coefficients of $(x+y)^n$?
   Solution: No one correct solution. Student answers may vary based on personal observations. Lecture 3 will discuss in detail what patterns we care about for this activity. Any observations however are correct.
   (b) How do the coefficients of $(x+y)^n$ compare to the coefficients of $(x+y)^{n+1}$?
   Solution: As in part (a) there is no one correct solution. Some observations regarding values being higher, or there being more coefficients could be made. Students may recall they sum to 2^n and 2^{n+1} respectively. Some students may even find patters that will be examined in more depth in lecture 3.
   (c) How many different terms does $(x+y)^n$ have for any n? Why do you think it has this many terms?
   Solution: There are $n+1$ terms. This is because we expand the $(x+y)$ n times and at each step we have a x or y to multiply by. We thus can choose $0, 1, ..., n$ number of $x's$ to build a term, creating $n+1$ terms.
   (d) Can you explain how the number of terms changes as n changes? Can you write down an argument for why this is true?
   Solution: Using part (c) we know that each one has $n+1$ terms so the number of terms increases linearly with n. This makes sense because if we increase the exponent by one, then we increase the number of times we multiply by $(x+y)$ by one and thus each term has exactly one more choice of x or y in its product.