The Strategy

Let the two players be Player 1 and Player 2.

You likely noticed that the player that brings the number of paper clips in the pile to 1 is guaranteed to win the game, and the player that brings the number of paper clips in the pile to 2, 3, or 4 generally loses the game. This is because the next player can bring the pile to 1 paper clip by removing 1, 2 or 3 paper clips, respectively.

Using similar reasoning, we can show that the player that brings the number of paper clips to 5 is guaranteed to be able to bring the number to 1 on their next turn, and the player that brings the number of paper clips to 9 is guaranteed to be able to bring the number to 5 on their next turn. This means that Player 1 has a winning strategy for this game and it goes as follows:

Start by removing 1 paper clip, reducing the total number of paper clips to 9. On your next turn, remove whatever number of paper clips are needed to bring the total to 5. On your turn after that, remove whatever number of paper clips are needed to bring the total to 1. (Our analysis above explains why each of these moves will be possible within the rules of the game.)

Notice that the “target numbers” (9, 5, and 1) all differ by 4. We can instead describe the strategy as follows: Go first and start by removing 1 paper clip. For all turns that follow, if the other player just removed $n$ paper clips, then you remove $4 - n$ paper clips. (These two turns, combined, will reduce the number of paper clips by 4.)

The Variations

Variation A

The player that reduces the pile to 1, 2, or 3 paper clips will lose the game since the next player can remove all of the remaining paper clips. Therefore, you want to be the player that reduces the pile to 4 paper clips as you are guaranteed to be able to win the game on your next turn. Player 1 now has the following winning strategy: Start by removing 2 paper clips, reducing the pile to 8 paper clips. On your next turn, remove whatever number of paper clips are needed to bring the total to 4. On your turn after that, remove all remaining paper clips.

Variations B and C

In each of these variations, players can remove 1, 3, or 4 paper clips on their turn, and you win by removing the last paper clip from the pile. Player 1 has a winning strategy starting from 10 paper clips (Variation B) and Player 2 has a winning strategy starting from 14 paper clips (Variation C). We outline these strategies in the table on the next page by analyzing how to win the game starting with each of 1 through 14 paper clips, in turn. We give the first move(s) in each strategy and then give guidance on how to use earlier rows in the table to fill in the rest of the strategy.
Let the two players be Ally and Bri. In each game, Ally will go first.

<table>
<thead>
<tr>
<th>Starting Pile</th>
<th>Winner</th>
<th>Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Player 1</td>
<td>Ally takes the clip and wins.</td>
</tr>
<tr>
<td>2</td>
<td>Player 2</td>
<td>Ally must take 1 clip. Bri takes the remaining clip and wins.</td>
</tr>
<tr>
<td>3</td>
<td>Player 1</td>
<td>Ally takes all 3 clips and wins.</td>
</tr>
<tr>
<td>4</td>
<td>Player 1</td>
<td>Ally takes all 4 clips and wins.</td>
</tr>
</tbody>
</table>
| 5             | Player 1 | Ally takes 3 clips, leaving 2 clips for Bri’s turn.  
As above, if a pile has 2 clips, the second player will win.  
Since it is Bri’s turn, the second player (starting from 2 clips) is Ally (Player 1). |
| 6             | Player 1 | Ally takes 4 clips, leaving 2 clips for Bri’s turn.  
As above, if a pile has 2 clips, the second player will win.  
Since it is Bri’s turn, the second player (starting from 2 clips) is Ally (Player 1). |
| 7             | Player 2 | Ally must take 1, 3, or 4 clips, leaving 6, 4, or 3 clips for Bri’s turn.  
As above, if a pile has 6, 4, or 3 clips, the first player will win.  
Since it is Bri’s turn, the first player is Bri (Player 2). |
| 8             | Player 1 | Ally takes 1 clip, leaving 7 clips for Bri’s turn.  
As above, if a pile has 7 clips, the second player will win.  
Since it is Bri’s turn, the second player is Ally (Player 1). |
| 9             | Player 2 | Ally must take 1, 3, or 4 clips, leaving 8, 6, or 5 clips for Bri’s turn.  
As above, if a pile has 8, 6, or 5 clips, the first player will win.  
Since it is Bri’s turn, the first player is Bri (Player 2). |
| 10            | Player 1 | Ally takes 1 clip, leaving 9 clips for Bri’s turn.  
As above, if a pile has 9 clips, the second player will win.  
Since it is Bri’s turn, the second player is Ally (Player 1). |
| 11            | Player 1 | Ally takes 4 clips, leaving 7 clips for Bri’s turn.  
As above, if a pile has 7 clips, the second player will win.  
Since it is Bri’s turn, the second player is Ally (Player 1). |
| 12            | Player 1 | Ally takes 3 clips, leaving 9 clips for Bri’s turn.  
As above, if a pile has 9 clips, the second player will win.  
Since it is Bri’s turn, the second player is Ally (Player 1). |
| 13            | Player 1 | Ally takes 4 clips, leaving 9 clips for Bri’s turn.  
As above, if a pile has 9 clips, the second player will win.  
Since it is Bri’s turn, the second player is Ally (Player 1). |
| 14            | Player 2 | Ally must take 1, 3, or 4 clips, leaving 13, 11, or 10 clips for Bri’s turn.  
As above, if a pile has 13, 11, or 10 clips, the first player will win.  
Since it is Bri’s turn, the first player is Bri (Player 2). |