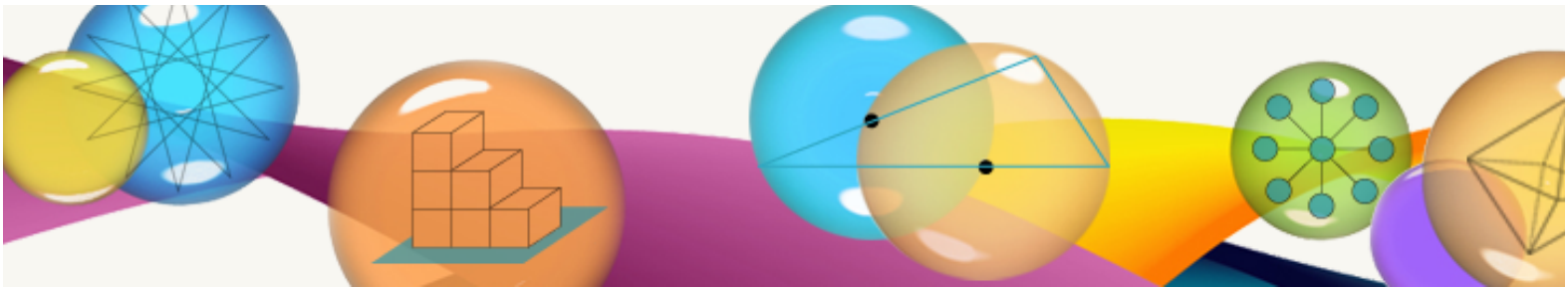


Problème de la semaine
9e/10e année (D)

Problèmes et solutions

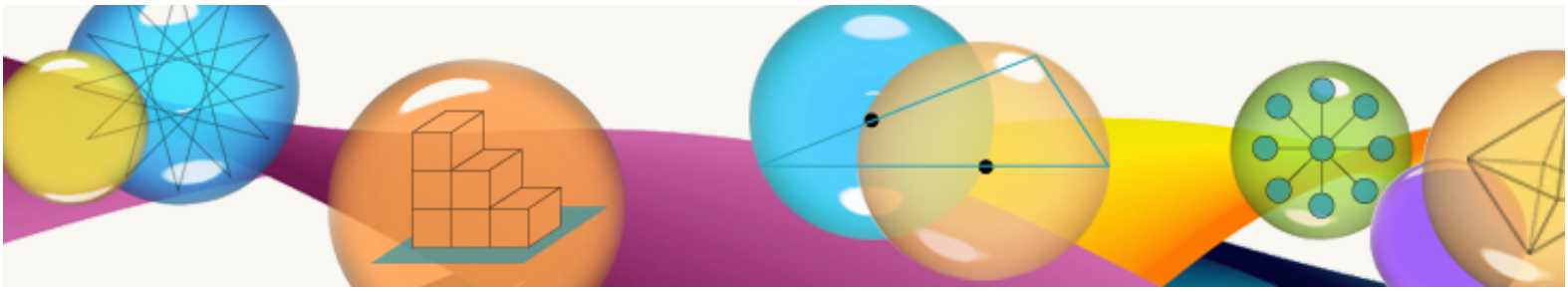
(solutions disponibles en anglais seulement)

2017 - 2018

Domaines

(Cliquer sur le nom du domaine ci-dessus pour sauter à cette section)

Les problèmes dans ce livret sont organisés par domaines. Un problème peut apparaître dans plusieurs domaines. Les problèmes sont appropriés pour la plupart des étudiants de 9e année ou plus avancés.



Traitement de données et probabilité

Amène-moi à la couverture

Problème de la semaine

Problème D

Répé-Trio

Trois balles de tennis, numérotés 1, 2 et 3, sont placées dans un sac. Une balle est retirée du sac et le numéro est noté. La balle est ensuite remise dans le sac.

Après avoir répété trois fois ce processus, quelle est la probabilité que la somme des trois nombres ainsi notés soit inférieure à 8?

Problem of the Week
Problem D and Solution
A Three-peat

Problem

Three tennis balls numbered 1, 2, and 3 are placed in a bag. A ball is drawn from the bag and the number is recorded. The ball is returned to the bag. After this is done three times, what is the probability that the sum of the three recorded numbers is less than 8?

Solution

In order to determine the probability, we must determine the number of ways three balls whose sum is less than 8 can be drawn from the bag and divide by the total number of ways three balls can be drawn from the bag.

First, let's determine the total number of ways three balls can be drawn from the bag. The balls are replaced after each draw so each time a ball is drawn from the bag it could be a 1, 2 or 3. Since three draws are made and there are three possible outcomes per draw, there are $3 \times 3 \times 3 = 27$ possible ways to draw three balls from the bag.

In Solution 1, we take a direct approach to counting the number of ways a sum of less than 8 can be obtained. In Solution 2, our approach is indirect. We count the number of ways a sum of 8 or more can be obtained and subtract this number from 27 to obtain the desired sum. In this problem it is actually easier to count the desired sum in this indirect way.

Solution 1

Let's determine how many of the 27 draws result in a sum that is less than 8 by systematically looking at the possible selections.

- Ball 1 is drawn three times. In this case the sum will be $1 + 1 + 1 = 3 < 8$. This can be done only 1 way: $1, 1, 1$.
- Ball 1 is drawn twice and ball 2 is drawn once. In this case the sum will be $1 + 1 + 2 = 4 < 8$. This can be done 3 ways: $1, 1, 2$ or $1, 2, 1$ or $2, 1, 1$.
- Ball 1 is drawn twice and ball 3 is drawn once. In this case the sum will be $1 + 1 + 3 = 5 < 8$. This can be done 3 ways: $1, 1, 3$ or $1, 3, 1$ or $3, 1, 1$.
- Ball 1 is drawn once and ball 2 is drawn twice. In this case the sum will be $1 + 2 + 2 = 5 < 8$. This can be done 3 ways: $1, 2, 2$ or $2, 1, 2$ or $2, 2, 1$.
- Ball 1 is drawn once and ball 3 is drawn twice. In this case the sum will be $1 + 3 + 3 = 7 < 8$. This can be done 3 ways: $1, 3, 3$ or $3, 1, 3$ or $3, 3, 1$.

- Ball 1 is drawn once, ball 2 is drawn once and ball 3 is drawn once. In this case the sum will be $1+2+3=6<8$. This can be done 6 ways: $1,2,3$ or $1,3,2$ or $2,1,3$ or $2,3,1$ or $3,1,2$ or $3,2,1$.
- Ball 2 is drawn three times. In this case the sum will be $2+2+2=6<8$. This can be done only 1 way: $2,2,2$.
- Ball 2 is drawn twice and ball 3 is drawn once. In this case the sum will be $2+2+3=7<8$. This can be done 3 ways: $2,2,3$ or $2,3,2$ or $3,2,2$.
- Ball 2 is drawn once and ball 3 is drawn twice. In this case the sum will be $2+3+3=8$, which is not less than 8.
- Ball 3 is drawn three times. In this case the sum will be $3+3+3=9$, which is not less than 8.

We see that there are $1+3+3+3+6+3+1+3=23$ ways to draw the balls so that the sum of the recorded numbers is less than 8.

Therefore, the probability that the sum is less than 8 is $\frac{23}{27}$.

Solution 2

Let's determine how many of the 27 draws result in a sum that is 8 or more. Since the maximum sum is 9, we need to count the number of ways the sum is 8 or 9.

- The sum is 8. The only way to do this is to draw ball 2 once and ball 3 twice. This can be done 3 ways: $2,3,3$ or $3,2,3$ or $3,3,2$.
- The sum is 9. The only way to do this is to draw ball 3 three times.

We see that there are $3+1=4$ ways to draw the balls so that the recorded sum is 8 or 9. Therefore, of the 27 outcomes, $27-4=23$ give a sum less than 8.

Therefore, the probability that the sum is less than 8 is $\frac{23}{27}$.

The indirect approach used in the second solution is definitely more efficient!

For Further Thought:

If this were a game, it would be *unfair* since the probability of obtaining a sum less than 8 is $\frac{23}{27}$ or 85% while the probability of obtaining a sum of 8 or higher is $\frac{4}{27}$ or 15%. In a fair game, we want the chance of two different outcomes occurring to be the same. Can you create a fair game out of this problem?

Problème de la semaine

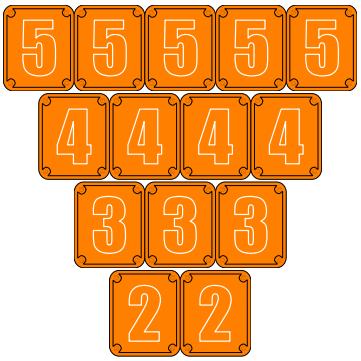
Problème D

Adresse sélective

Une quincaillerie vend des numéros individuels pour indiquer l'adresse d'une résidence. Elle a en inventaire cinq numéros 5, quatre numéros 4, trois numéros 3 et deux numéros 2.

À partir de cet assortiment, une cliente peut acheter les trois numéros dont elle a besoin pour l'adresse de sa maison.

Combien de combinaisons différentes de nombres à trois chiffres pourrait-elle créer?



Problem of the Week
Problem D and Solution
Select Addresses

Problem

A hardware store sells single digits to be used for house numbers. There are five 5s, four 4s, three 3s, and two 2s available. From this selection of digits, a customer is able to purchase her three-digit house number. Determine the number of possible three-digit house numbers this customer could form.

Solution

Solution 1

Let's suppose that there are three or more 2s available.
For the first digit, the customer can choose from the digits 5, 4, 3, and 2. Therefore, there are 4 choices for the first digit.
Similarly, there are 4 choices for the second digit and 4 choices for the third digit.
This gives $4 \times 4 \times 4 = 64$ possible three-digit house numbers that can be made.
But there are actually only two 2s available, so not all of these house numbers can be made. In particular, the house number 222 cannot be made, but all others can.
Therefore, the customer could form $64 - 1 = 63$ different three-digit house numbers.

Solution 2

Let's look at the different cases.

<u>Case 1:</u> All three digits in the house number are the same
The house number could then be 555, 444 or 333. The customer cannot form 222 since only two 2s are available. Therefore, there are 3 three-digit house numbers with all three digits the same.

<u>Case 2:</u> Two digits are the same and the third digit is different
There are four choices for the digits that are the same, namely 5, 4, 3, and 2. For each of these possible choices, there are 3 choices for the third different digit. For example, if two of the digits are 5s, then the third digit could be 4, 3 or 2. Therefore, there are $4 \times 3 = 12$ ways to choose the digits. For each of these choices, there are 3 ways to arrange the digits. For example, suppose the digits are a, a and b. The house number could be aab, aba or baa. Therefore, there are $12 \times 3 = 36$ three-digit house numbers with two digits the same and one different.

<u>Case 3:</u> All three digits are different
The customer has 4 choices for the first digit (namely, 5, 4, 3, or 2). Once that digit is chosen, there are 3 choices for the second digit. Once the first and second digits are chosen, there are 2 choices for the third digit. Therefore, there are $4 \times 3 \times 2 = 24$ three-digit house numbers with all three digits different.

Therefore, the total number of three-digit house numbers that the customer can form is $3 + 36 + 24 = 63$.

Problème de la semaine
Problème D
Sport en famille

Cinq paires, chaque paire étant composée d'un frère et d'une sœur, participent aux Jeux olympiques d'hiver. Les membres d'une même famille participent aux épreuves du même sport. Ces paires "frère-et-sœur" participent à des sports différents.

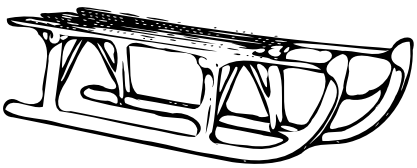
En utilisant les indices suivants, détermine les paires "frère-et-sœur" ainsi que leur sport.

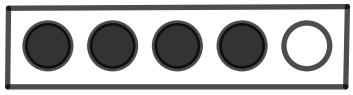
1. Caleb est le frère de Grâce.
2. Isabelle ne participe pas à la compétition de luge.
3. Anton participe au biathlon et n'est pas le frère d'Isabelle.
4. Jess participe à l'épreuve de slalom et Daniel n'est pas son frère.
5. Ezra et sa sœur Helga participent à lépreuve de ski acrobatique (bosses).

Le tableau suivant pourrait t'aider à résoudre ce problème.

	Frieda	Grâce	Helga	Isabelle	Jess	Biathlon	Curling	Luge	Bosses	Slalom
Anton										
Brian										
Caleb										
Daniel										
Ezra										

Biathlon					
Curling					
Luge					
Bosses					
Slalom					





Problem of the Week
Problem D and Solution
All in the Family

Problem

Five brother-sister pairs are participating at the Winter Olympics. Each pair is involved in the events in the same sport. No two pairs are in the same sport. Using the following clues, determine each brother-sister pair and their sport.

1. Grace's brother is Caleb.
2. Jess participates in Slalom and her brother is not Daniel.
3. Isabelle is not a participant in Luge at the Olympics.
4. Ezra and his sister Helga are participating in Moguls.
5. Anton, in the Biathlon, is not Isabelle's brother.

Solution

Using the ordered triplets (sister, brother, sport) the solution is; (Frieda, Anton, Biathlon), (Grace, Caleb, Luge), (Helga, Ezra, Moguls), (Isabelle, Daniel, Curling) and (Jess, Brian, Slalom). A possible solution is shown below.

In our solution, we will we go through each clue and update the table based on the information in the clue. We will put an X in a cell if the combination indicated by the row and column for that cell is not possible, or a ✓ if it must be true.

In clue (1) we are told Grace and Caleb are brother and sister. We will put a ✓ in the corresponding cell. We then put X's in the cells corresponding to Frieda, Helga, Isabelle and Jess being Caleb's sister and in the cells corresponding to Anton, Brian, Daniel and Ezra being Grace's brother.
In clue (2), we are told Jess participates in Slalom. We will put a ✓ in the cell corresponding to Jess and Slalom. We then put X's in the cells corresponding to Jess participating in Biathlon, Curling, Luge or Moguls and for each cell corresponding to Freida, Grace, Helga or Isabelle participating in Slalom. Also Jess is not Daniel's brother. So we will put an X in corresponding cell. Since Jess does Slalom and Daniel is not her brother, therefore Daniel cannot participate in Slalom. So we will put an X in the corresponding cell. These are shown on the chart below.

	Frieda	Grace	Helga	Isabelle	Jess	Biathlon	Curling	Luge	Moguls	Slalom
Anton		X								
Brian		X								
Caleb	X	✓	X	X	X					
Daniel		X			X					X
Ezra		X								
Biathlon					X					
Curling					X					
Luge					X					
Moguls					X					
Slalom	X	X	X	X	✓					

In clue (3), Isabelle does not participate in Luge, we put an X in the corresponding cell.
In clue (4) we are told Ezra and Helga are brother and sister. We will put a ✓ in the corresponding cell. We then put X's in the cells corresponding to Frieda, Isabelle orJess being Ezra's sister and in the cells corresponding to Anton, Brian, or Daniel being Helga's brother. We are also told Ezra participates in Moguls. We will put a ✓ in the corresponding cell. We then put X's in the cells corresponding to Ezra participating in Biathlon, Curling, Luge or Slalom and in the cells corresponding to Anton, Brian, Caleb or Daniel participating in Moguls. We are also told Helga participates in Moguls. We will put a ✓ in the corresponding cell. We then put X's in the cells corresponding to Helga participating in Biathlon, Curling, or Luge and in the cells corresponding to Frieda, Grace or Isabelle participating in Moguls.The updated chart is shown below.

	Frieda	Grace	Helga	Isabelle	Jess	Biathlon	Curling	Luge	Moguls	Slalom
Anton		X	X						X	
Brian		X	X						X	
Caleb	X	✓	X	X	X				X	
Daniel		X	X		X				X	X
Ezra	X	X	✓	X	X	X	X	X	✓	X
Biathlon			X		X					
Curling			X		X					
Luge			X	X	X					
Moguls	X	X	✓	X	X					
Slalom	X	X	X	X	✓					

In clue (5), we are told Anton participates in Biathlon. We will put a ✓ in the corresponding cell. We then put X's in the cells corresponding to Anton participating in Curling, Luge or Slalom and in the cells corresponding to Brian, Caleb or Daniel participating in Biathlon. We are also told that Anton is not Isabelle's brother. We will put an X in the corresponding cell. Also since Anton does Biathlon and is not Isabelle's brother, then Isabelle does not do biathlon and we will put an X the corresponding cell. The updated chart is shown below.

	Frieda	Grace	Helga	Isabelle	Jess	Biathlon	Curling	Luge	Moguls	Slalom
Anton		X	X	X		✓	X	X	X	X
Brian		X	X			X			X	
Caleb	X	✓	X	X	X	X			X	
Daniel		X	X		X	X			X	X
Ezra	X	X	✓	X	X	X	X	X	✓	X
Biathlon			X	X	X					
Curling			X		X					
Luge			X	X	X					
Moguls	X	X	✓	X	X					
Slalom	X	X	X	X	✓					

Now if you look at Isabelle's column you will notice that Curling is the only sports option left for Isabelle. We will put a ✓ in the corresponding cell and put X's in the cells corresponding to Frieda or Grace participating in Curling. Also notice that Anton is in Biathlon and Jess is in Slalom so they are not brother and sister. So we will put an X in the corresponding cell. The updated chart is shown below.

	Frieda	Grace	Helga	Isabelle	Jess	Biathlon	Curling	Luge	Moguls	Slalom
Anton		X	X	X	X	✓	X	X	X	X
Brian		X	X			X			X	
Caleb	X	✓	X	X	X	X			X	
Daniel		X	X		X	X			X	X
Ezra	X	X	✓	X	X	X	X	X	✓	X
Biathlon			X	X	X					
Curling	X	X	X	✓	X					
Luge			X	X	X					
Moguls	X	X	✓	X	X					
Slalom	X	X	X	X	✓					

Now, if you look at Anton's row, you will notice that Frieda is the only sister option for Anton. We will put a ✓ in the corresponding cell and put X's in the cells corresponding to Frieda being sister to Brian or Daniel. If you look at Jess's column you will notice that Brian is the only brother option for Jess. We will put a ✓ in the corresponding cell and put an X in the cell corresponding to Isabelle being a sister to Brian. The updated chart is shown below.

	Frieda	Grace	Helga	Isabelle	Jess	Biathlon	Curling	Luge	Moguls	Slalom
Anton	✓	X	X	X	X	✓	X	X	X	X
Brian	X	X	X	X	✓	X			X	
Caleb	X	✓	X	X	X	X			X	
Daniel	X	X	X		X	X			X	X
Ezra	X	X	✓	X	X	X	X	X	✓	X
Biathlon			X	X	X					
Curling	X	X	X	✓	X					
Luge			X	X	X					
Moguls	X	X	✓	X	X					
Slalom	X	X	X	X	✓					

Now, if you look at Daniel's row, you will notice that Isabelle is the only sister option for Daniel. We will put a ✓ in the corresponding cell. Now Anton participates in Biathlon and Anton is Frieda's brother, therefore Frieda also participates in Biathlon. We will put a ✓ in the corresponding cell and put an X in the cell corresponding to Grace corresponding to Biathlon and in the cell corresponding to Frieda participating in Luge. Also Jess is Brian's sister and Jess in the Slalom, therefore Brian is in the Slalom. We will put a ✓ in the corresponding cell and put an X in the cell corresponding to Caleb participating in Slalom and X's in the cells corresponding to Brian participating in Curling or luge. The updated chart is shown below.

	Frieda	Grace	Helga	Isabelle	Jess	Biathlon	Curling	Luge	Moguls	Slalom
Anton	✓	X	X	X	X	✓	X	X	X	X
Brian	X	X	X	X	✓	X	X	X	X	✓
Caleb	X	✓	X	X	X	X			X	X
Daniel	X	X	X	✓	X	X			X	X
Ezra	X	X	✓	X	X	X	X	X	✓	X
Biathlon	✓	X	X	X	X					
Curling	X	X	X	✓	X					
Luge	X		X	X	X					
Moguls	X	X	✓	X	X					
Slalom	X	X	X	X	✓					

Finally, if you look at Grace's column, Luge is the only option for a sport. So we will put a ✓ in the corresponding cell. Since Grace is in Luge and Caleb is her brother then Caleb is in luge, therefore we will put a ✓ in the corresponding cell. Also note that Isabelle is in Curling and Daniel is her brother, therefore Daniel is in Curling. We will put a ✓ in the corresponding cell. We now put X's in the cell corresponding to Caleb participating in Curling and in the cell corresponding to Daniel participating in Luge. The final updated chart is shown below.

	Frieda	Grace	Helga	Isabelle	Jess	Biathlon	Curling	Luge	Moguls	Slalom
Anton	✓	X	X	X	X	✓	X	X	X	X
Brian	X	X	X	X	✓	X	X	X	X	✓
Caleb	X	✓	X	X	X	X	X	✓	X	X
Daniel	X	X	X	✓	X	X	✓	X	X	X
Ezra	X	X	✓	X	X	X	X	X	✓	X
Biathlon	✓	X	X	X	X					
Curling	X	X	X	✓	X					
Luge	X	✓	X	X	X					
Moguls	X	X	✓	X	X					
Slalom	X	X	X	X	✓					

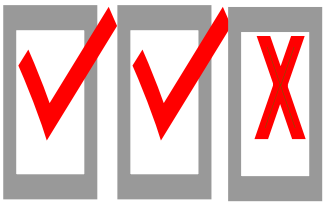
Problème de la semaine

Problème D

Pourcentage d'intérêt

L'information suivante est connue au sujet des résultats d'un test de mathématiques :

- le test possédait trois questions;
- chaque question avait une valeur de 1 point;
- chaque réponse était évaluée, soit correcte ou incorrecte;
- 50% des élèves ont eu les 3 bonnes réponses;
- 5% des élèves n'ont pas eu de bonne réponse; et
- la note moyenne de la classe était de 2,3 sur 3.



Détermine le pourcentage d'élèves qui ont eu exactement une bonne réponse ainsi que le pourcentage d'élève qui ont exactement 2 bonnes réponses.

Problem of the Week
Problem D and Solution
Percents of Interest

Problem

The following information is known about the results of a recent math test: there were three questions on the test, each question was worth 1 mark, each question was marked right or wrong (no part marks), 50% of the students got all 3 questions correct, 5% of the students got no questions correct, and the class average mark was 2.3 out of 3. Determine the percentage of students who got exactly 1 question correct and the percentage of students who got exactly 2 questions correct.

Solution

Solution 1

In this solution, we will use only one variable.

To determine an average, we must determine the total of all the scores and divide by the number of students. Without changing the overall class average, suppose that 100 students wrote this test.

Let x represent the percent who got exactly 2 questions correct.
Then $100 - 50 - 5 - x = (45 - x)$ percent got exactly 1 question correct.

Since 50% of the students got all 3 questions correct, 50 students each scored 3 marks and earned a total of $50 \times 3 = 150$ marks.

Since x% of the students got exactly 2 questions correct, x students each scored 2 marks and earned a total of $x \times 2 = 2x$ marks.

Since $(45 - x)$% of the students got exactly 1 question correct, $(45 - x)$ students each scored 1 mark and earned a total of $(45 - x) \times 1 = (45 - x)$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

The total number of marks earned by the 100 students was $150 + 2x + (45 - x) + 0 = x + 195$.

We know that the average score was 2.3, so

$$\begin{aligned} \frac{x + 195}{100} &= 2.3 \\ x + 195 &= 230 \\ x &= 35 \\ 45 - x &= 10 \end{aligned}$$

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.

Solution 2

In this solution, we will use two variables.

To determine an average, we must determine the total of all the scores and divide by the number of students. Without changing the overall class average, suppose that 100 students wrote this test.

Let x represent the percent who got exactly 2 questions correct.
Let y represent the percent who got exactly 1 question correct.

Then, $50 + x + y + 5 = 100$ which simplifies to $x + y = 45$. (1)

Since 50% of the students got all 3 questions correct, 50 students each scored 3 marks and earned a total of $50 \times 3 = 150$ marks.

Since x% of the students got exactly 2 questions correct, x students each scored 2 marks and earned a total of $x \times 2 = 2x$ marks.

Since y% of the students got exactly 1 question correct, y students each scored 1 mark and earned a total of $y \times 1 = y$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

The total number of marks earned by the 100 students was $150 + 2x + y + 0 = 2x + y + 150$.

We know that the average score was 2.3, so

$$\begin{aligned} \frac{2x + y + 150}{100} &= 2.3 \\ 2x + y + 150 &= 230 \\ 2x + y &= 80 \quad (2) \end{aligned}$$

Subtracting equation (1) from equation (2), we obtain $x = 35$. Substituting $x = 35$ into equation (1), we obtain $y = 10$.

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.

Solution 3

In this solution, we will use 2 variables but we will not assume a class size.

To determine an average, we must determine the total of all the scores and divide by the number of students.

Let n represent the number of students who wrote the test where n is a positive integer.

Let x represent the percent who got exactly 2 questions correct.
Let y represent the percent who got exactly 1 question correct.

Then, $50 + x + y + 5 = 100$ which simplifies to $x + y = 45$. (1)

Since 50% of the students got all 3 questions correct, $\frac{50}{100}n$ students each scored 3 marks and earned a total of $\frac{50}{100}n \times 3 = \frac{150n}{100}$ marks.

Since x% of the students got exactly 2 questions correct, $\frac{x}{100}n$ students each scored 2 marks and earned a total of $\frac{x}{100}n \times 2 = \frac{2xn}{100}$ marks.

Since y% of the students got exactly 1 question correct, $\frac{y}{100}n$ students each scored 1 mark and earned a total of $\frac{y}{100}n \times 1 = \frac{yn}{100}$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

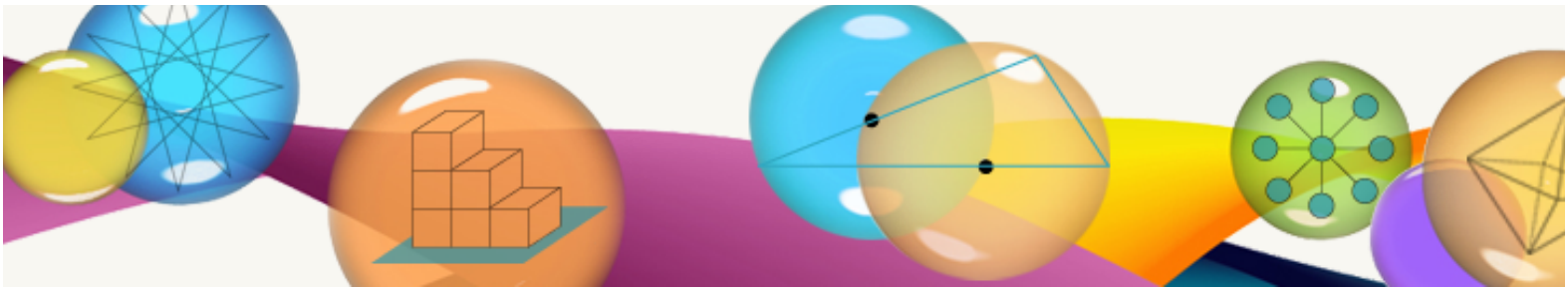
The total number of marks earned by the n students was $\frac{150n}{100} + \frac{2xn}{100} + \frac{yn}{100} = \frac{n}{100}(2x + y + 150)$.

We know that the average score was 2.3 and n is a positive integer, so

$$\begin{aligned} \frac{\frac{n}{100}(2x + y + 150)}{n} &= 2.3 \\ 2x + y + 150 &= 230 \\ 2x + y &= 80 \quad (2) \end{aligned}$$

Subtracting equation (1) from equation (2), we obtain $x = 35$. Substituting $x = 35$ into equation (1), we obtain $y = 10$.

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.



Mesure et trigonométrie

Amène-moi à la couverture

Problème de la semaine

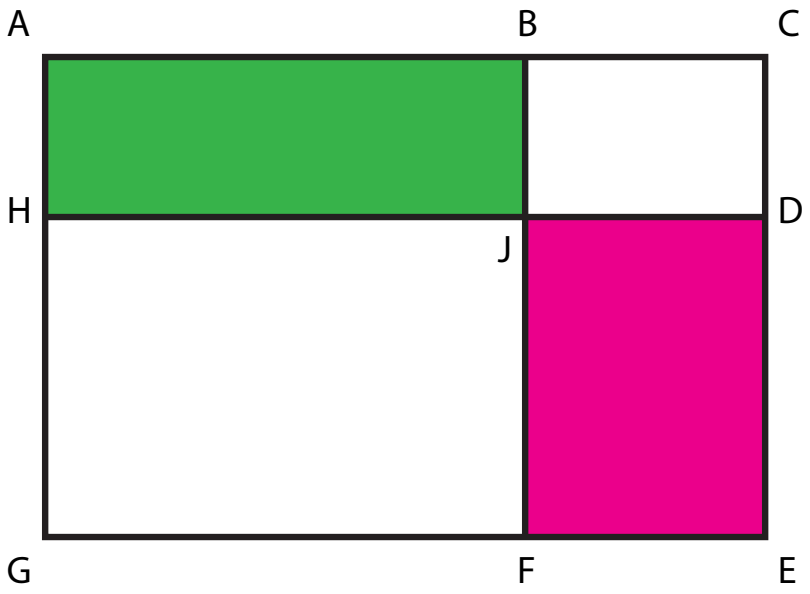
Problème D

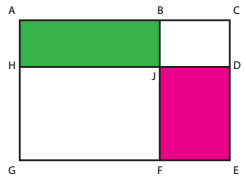
Maximiser l'aire

Les rectangles $ABJH$ et $JDEF$, dont la longueur de chacun de leurs côtés est un nombre entier, se touchent au coin J de façon à ce que le segment de droite HJD soit perpendiculaire au segment de droite BJF. Les deux rectangles sont inclus dans un plus grand rectangle $ACEG$, comme indiqué dans le diagramme ci-dessous.

L'aire du rectangle $ABJH$ est de 6 cm^2 et l'aire du rectangle $JDEF$ est de 15 cm^2.

Détermine la plus grande aire possible du rectangle $ACEG$. Il est à noter que le diagramme n'est pas à l'échelle.





Problem of the Week

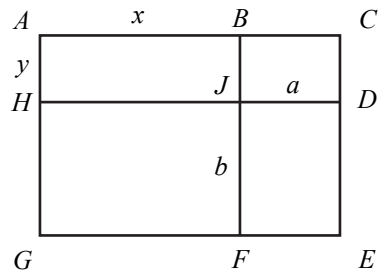
Problem D and Solution

Maximize the Area

Problem

Two rectangles, $ABJH$ and $JDEF$, with integer side lengths, share a common corner at J such that HJD and BJF are perpendicular line segments. The two rectangles are enclosed by a larger rectangle $ACEG$, as shown. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2. Determine the largest possible area of the rectangle $ACEG$. Note that the diagram is not intended to be to scale.

Solution



Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$.

Therefore,
$$\begin{aligned} AB &= HJ = GF = x, \\ AH &= BJ = CD = y, \\ BC &= JD = FE = a, \text{ and} \\ HG &= JF = DE = b. \end{aligned}$$

$$\begin{aligned} \text{Then area}(ACEG) &= \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG) \\ &= 6 + ya + 15 + xb \\ &= 21 + ya + xb \end{aligned}$$

Since the area of rectangle $ABJH$ is 6 cm^2 and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

Since the area of rectangle $JDEF$ is 15 cm^2 and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

To maximize the area, we need to pick the values of x, y, a, b which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of a and b. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of a and b. (As an extension, we will leave it to you to think about why this is the case.)

Case 1: $x = 1$ cm, $y = 6$ cm and $a = 1$ cm, $b = 15$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42 \text{ cm}^2$$

Case 2: $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112 \text{ cm}^2$$

Case 3: $x = 1$ cm, $y = 6$ cm and $a = 3$ cm, $b = 5$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44 \text{ cm}^2$$

Case 4: $x = 1$ cm, $y = 6$ cm and $a = 5$ cm, $b = 3$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54 \text{ cm}^2$$

Case 5: $x = 2$ cm, $y = 3$ cm and $a = 1$, $b = 15$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54 \text{ cm}^2$$

Case 6: $x = 2$ cm, $y = 3$ cm and $a = 15$, $b = 1$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68 \text{ cm}^2$$

Case 7: $x = 2$ cm, $y = 3$ cm and $a = 3$, $b = 5$ cm

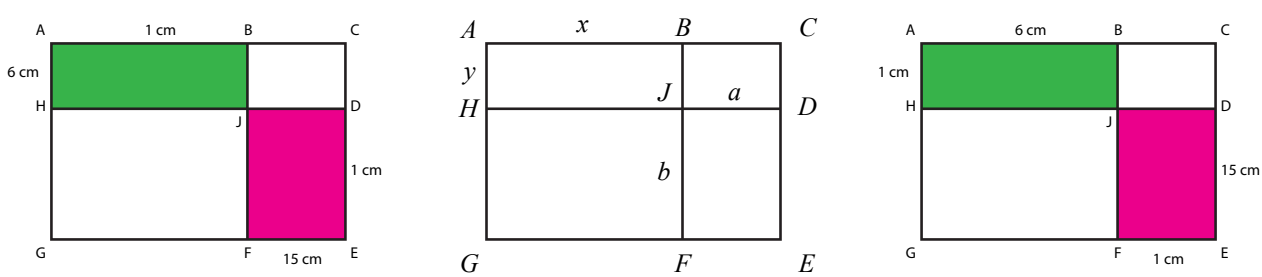
$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40 \text{ cm}^2$$

Case 8: $x = 2$ cm, $y = 3$ cm and $a = 5$, $b = 3$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42 \text{ cm}^2$$

We see that the maximum area is 112 cm^2, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm^2.



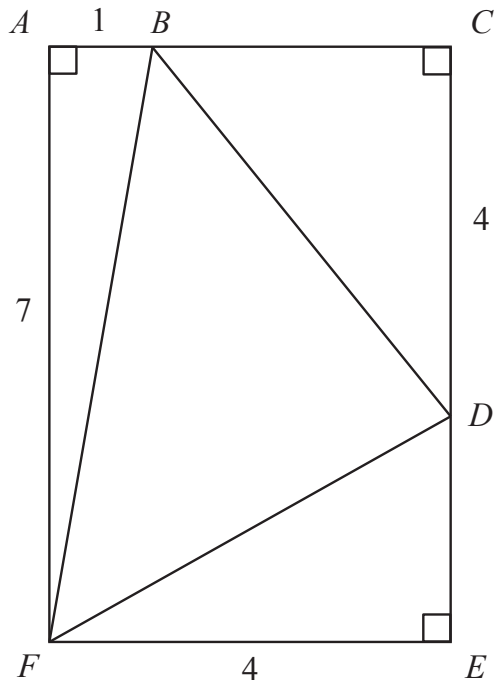
Problème de la semaine

Problème D

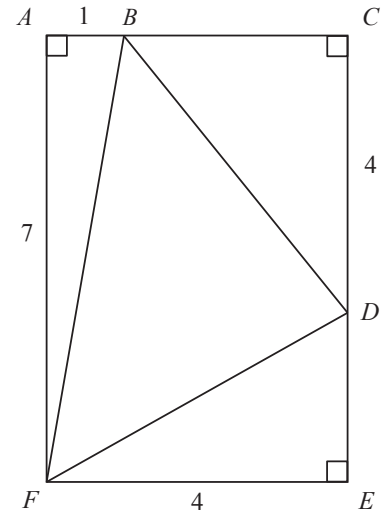
Plusieurs solutions

$ACEF$ est un rectangle dans lequel $FE = 4$ et $FA = 7$. Le triangle $\triangle BDF$ est inscrit dans le rectangle $ACEF$ de sorte que le point B soit situé sur le segment AC et que $AB = 1$. On veut aussi que le point D soit situé sur le segment CE et que $CD = 4$.

Détermine la valeur de $\angle ABF + \angle CBD$.



Problem of the Week
Problem D and Solution
Multiple Solutions



Problem

$ACEF$ is a rectangle with $FE = 4$ and $FA = 7$. $\triangle BDF$ is inscribed in rectangle $ACEF$ with B on AC such that $AB = 1$ and D on CE such that $CD = 4$.

Determine the value of $\angle ABF + \angle CBD$.

Solution

Since $ACEF$ is a rectangle, $\angle A = \angle C = \angle E = 90^\circ$, $AC = FE$ and $AF = CE$.

Since $AC = FE$, we have $BC = 3$. Since $AF = CE$, we have $DE = 3$.

We will now present three different solutions. The first uses the Pythagorean Theorem, the second uses congruent triangles, and the third uses basic trigonometry.

<u>Solution 1</u>

Since $\triangle BCD$ has a right angle at C, we can apply the Pythagorean Theorem to find that $BD^2 = BC^2 + CD^2 = 3^2 + 4^2 = 25$. Therefore, $BD = 5$, since $BD > 0$.

Similarly, since $\triangle FDE$ has a right angle at E, we can apply the Pythagorean Theorem to find that $DF = 5$.

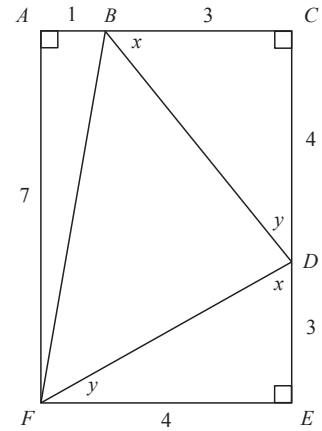
Since $\triangle ABF$ has a right angle at A, we can apply the Pythagorean Theroem to find that $BF^2 = AF^2 + AB^2 = 7^2 + 1^2 = 50$ and so $BF = \sqrt{50}$, since $BF > 0$.

In $\triangle BDF$, notice that $BD^2 + DF^2 = 5^2 + 5^2 = 25 + 25 = 50 = BF^2$. Therefore, $\triangle BDF$ is a right triangle, with $\angle BDF = 90^\circ$. Also, since $BD = DF = 5$, $\triangle BDF$ is an isosceles right triangle and so $\angle DBF = \angle DFB = 45^\circ$.

The angles in a straight line sum to 180°, so we have
$\angle ABF + \angle DBF + \angle CBD = 180^\circ$.
Since $\angle DBF = 45^\circ$, this becomes $\angle ABF + 45^\circ + \angle CBD = 180^\circ$, and so
$\angle ABF + \angle CBD = 180^\circ - 45^\circ = 135^\circ$.

Therefore, $\angle ABF + \angle CBD = 135^\circ$.

Solution 2



Looking at $\triangle BCD$ and $\triangle DEF$, we have $CD = FE = 4$, $DE = BC = 3$ and $\angle C = \angle E = 90^\circ$.

Therefore $\triangle BCD \cong \triangle DEF$ by side-angle-side triangle congruency. From the triangle congruency, it follows that $\angle DFE = \angle BDC = y$, $\angle EDF = \angle DBC = x$ and $BD = DF$.

Since the angles in a triangle sum to 180°, in right $\triangle BDC$, $\angle DBC + \angle BDC = 90^\circ$. But $\angle DBC = \angle EDF$. Substituting, we obtain $\angle EDF + \angle BDC = 90^\circ$. (1)

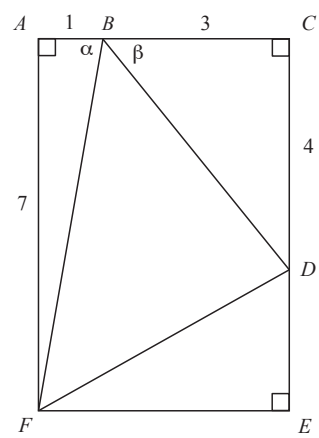
Since the angles in a straight line sum to 180°, $\angle BDC + \angle BDF + \angle EDF = 180^\circ$. But, from (1), $\angle EDF + \angle BDC = 90^\circ$. Substituting, we obtain $90^\circ + \angle BDF = 180^\circ$ and $\angle BDF = 90^\circ$ follows.

Since $BD = DF$ and $\angle BDF = 90^\circ$, $\triangle BDF$ is an isosceles right triangle and so $\angle DBF = \angle BFD = 45^\circ$.

The angles in a straight line sum to 180°, so we have $\angle ABF + \angle DBF + \angle CBD = 180^\circ$. Since $\angle DBF = 45^\circ$, this becomes $\angle ABF + 45^\circ + \angle CBD = 180^\circ$, and so $\angle ABF + \angle CBD = 180^\circ - 45^\circ = 135^\circ$.

Therefore, $\angle ABF + \angle CBD = 135^\circ$.

Solution 3



In right $\triangle ABF$, let $\angle ABF = \alpha$. Using basic trigonometry, $\tan\alpha = \frac{7}{1} = 7$ and $\alpha = \tan^{-1}(7)$.

In right triangle CBD, let $\angle CBD = \beta$. Again, using basic trigonometry, $\tan\beta = \frac{4}{3}$ and $\beta = \tan^{-1}\left(\frac{4}{3}\right)$.

Then $\angle ABF + \angle CBD = \alpha + \beta = \tan^{-1}(7) + \tan^{-1}\left(\frac{4}{3}\right) = 135^\circ$.

Therefore, $\angle ABF + \angle CBD = 135^\circ$.

This third solution is very efficient and concise. However, some of the beauty is lost as a result of the direct approach.

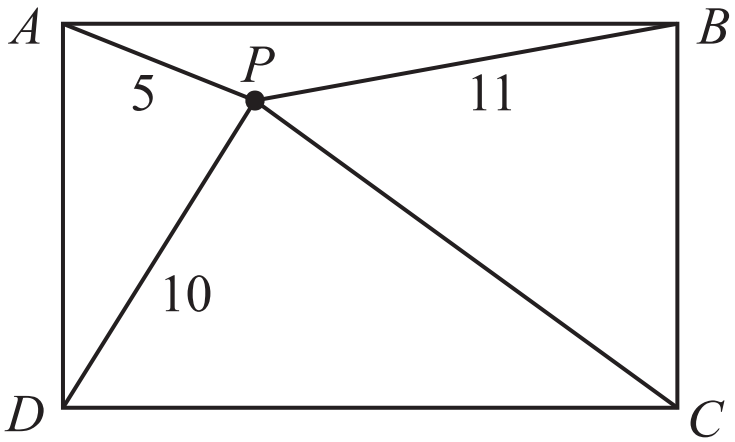
Problème de la semaine

Problème D

Aux quatre coins

Dans le diagramme ci-dessous, $ABCD$ est un rectangle. Le point P est situé à l'intérieur du rectangle de façon à ce que la distance entre P et A est de 5 cm, la distance entre P et B est de 11 cm, et la distance entre P et D est de 10 cm.

Quelle est la distance entre P et C?

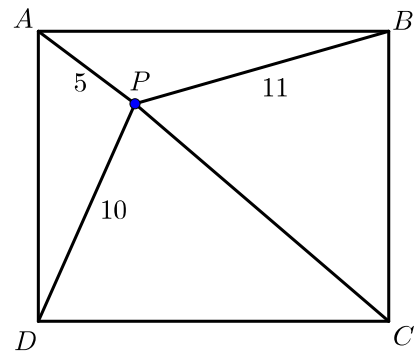


SUGGESTION : trace un segment de droite qui passe par le point P et qui est perpendiculaire à deux des côtés du rectangle. Utilise ensuite le théorème de Pythagore.

Problem of the Week

Problem D and Solution

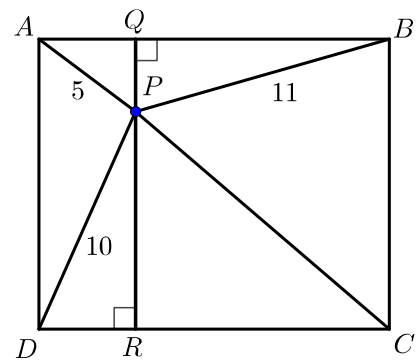
From the Four Corners



Problem

In the diagram, $ABCD$ is a rectangle. Point P is located inside the rectangle so that the distance from P to A is 5 cm, the distance from P to B is 11 cm, and the distance from P to D is 10 cm. How far is P from C?

Solution



We start by drawing a perpendicular from P to AB. Let Q be the point of intersection. Let's draw another perpendicular from P to DC. Let R be the point of intersection.

Since QP is perpendicular to AB, $\angle AQP = 90^\circ$ and $\angle BQP = 90^\circ$. Since PR is perpendicular to DC, $\angle DRP = 90^\circ$ and $\angle CRP = 90^\circ$. We also have that $AQ = DR$ and $BQ = CR$.

We can apply the Pythagorean Theorem in $\triangle AQP$ and $\triangle BQP$.
From $\triangle AQP$ we have $AQ^2 + QP^2 = AP^2$, and so $AQ^2 + QP^2 = 5^2 = 25$.
Rearranging, we have $QP^2 = 25 - AQ^2$ (1).

From $\triangle BQP$ we have $BQ^2 + QP^2 = BP^2$, and so $BQ^2 + QP^2 = 11^2 = 121$.

Rearranging, we have $QP^2 = 121 - BQ^2$ (2).

Since $QP^2 = QP^2$, from (1) and (2) we find that $25 - AQ^2 = 121 - BQ^2$ or $BQ^2 - AQ^2 = 96$.
Since $AQ = DR$ and $BQ = CR$, this also tells us $CR^2 - DR^2 = 96$ (3).

We can now apply the Pythagorean Theorem in $\triangle DRP$ and $\triangle CRP$.
From $\triangle DRP$ we have $DR^2 + RP^2 = DP^2$, and so $DR^2 + RP^2 = 10^2 = 100$.
Rearranging, we have $RP^2 = 100 - DR^2$ (4).

When we apply the Pythagorean Theorem to $\triangle CRP$ we have $CR^2 + RP^2 = CP^2$.
Rearranging, we have $RP^2 = CP^2 - CR^2$ (5).

Since $RP^2 = RP^2$, from (4) and (5) we find that $100 - DR^2 = CP^2 - CR^2$, or
$CR^2 - DR^2 = CP^2 - 100$ (6).

From (3), we have $CR^2 - DR^2 = 96$, so (6) becomes $96 = CP^2 - 100$ or $CP^2 = 196$.
Thus $CP = 14$, since $CP > 0$.

Therefore the distance from P to C is 14 cm.

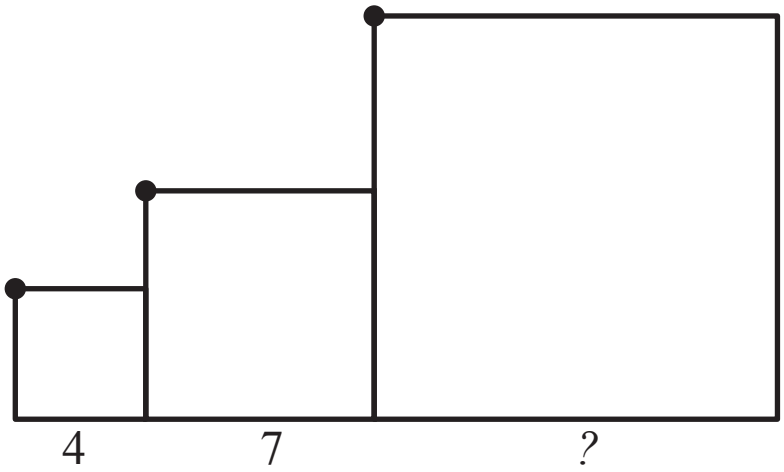
Problème de la semaine

Problème D

Grand, très grand, plus grand

Trois carrés sont placés l'un à côté de l'autre, comme le démontre le diagramme. Le côté du plus petit carré mesure 4 unités, celui du deuxième carré mesure 7 unités, mais la longueur du côté du plus grand carré est inconnue. Cependant, le coin supérieur gauche de chaque carré se trouve sur une même ligne droite.

Détermine la longueur du côté du plus grand carré.



Problem of the Week

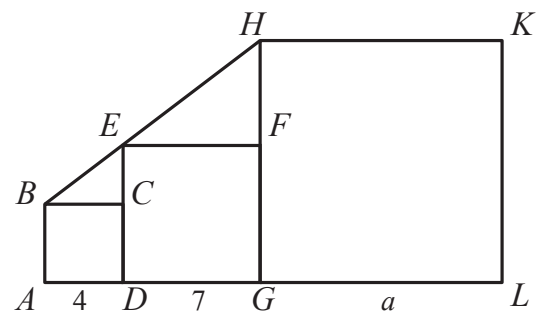
Problem D and Solution

Big, Bigger, Biggest

Problem

Three squares are placed beside each other as shown. The smallest square has side length 4 units, the middle-sized square has side length 7 units, but the side length of the largest square is unknown. However, the top left corner of each of the three squares lies on a straight line. Determine the side length of the largest square.

Solution



Label the vertices as shown on the diagram. Draw line segment BH through E. Let a represent the side length of the larger square.

In Solution 1, we will solve the problem by calculating the slope of BH.

In Solution 2, we will solve the problem using similar triangles.

In Solution 3, we will place the diagram on the xy-plane and solve the problem using analytic geometry.

Solution 1

The slope of a line is equal to its rise divided by its run.

If we look at the line segment from B to E, $BC = 4$ and $CE = DE - DC = 7 - 4 = 3$. Therefore, slope $BE = \frac{CE}{BC} = \frac{3}{4}$.

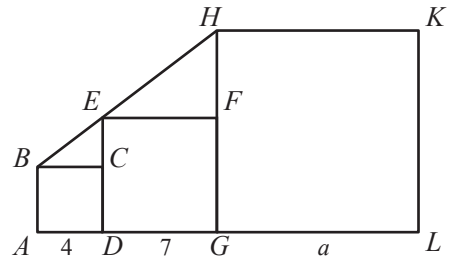
If we look at the line segment from E to H, $EF = 7$ and $FH = GH - GF = a - 7$. Therefore, slope $EH = \frac{FH}{EF} = \frac{a-7}{7}$.

Since B, E and H lie on a straight line, slope BE must equal slope EH. Therefore,

$$\begin{aligned} \text{slope } BE &= \text{slope } EH \\ \frac{3}{4} &= \frac{a-7}{7} \\ 4(a-7) &= 3(7) \\ 4a - 28 &= 21 \\ 4a &= 49 \end{aligned}$$

$\therefore a = \dfrac{49}{4}$ and the side length of the larger square is $\dfrac{49}{4}$ units.

Solution 2



Consider $\triangle BCE$ and $\triangle EFH$. We will first show that $\triangle BCE \sim \triangle EFH$.

Since $ABCD$ is a square, $\angle BCD = 90^\circ$.
Therefore, $\angle BCE = 180^\circ - \angle BCD = 180^\circ - 90^\circ = 90^\circ$.

Since $DEFG$ is a square, $\angle EFG = 90^\circ$.
Therefore, $\angle EFH = 180^\circ - \angle EFG = 180^\circ - 90^\circ = 90^\circ$.
Thus, $\angle BCE = \angle EFH$.

Since $ABCD$ and $DEFG$ are squares and AG is a straight line, BC is parallel to EF.
Therefore, $\angle EBC$ and $\angle HEF$ are corresponding angles and so $\angle EBC = \angle HEF$.

Since the angles in a triangle add to 180°, then we must also have $\angle BEC = \angle EHF$.

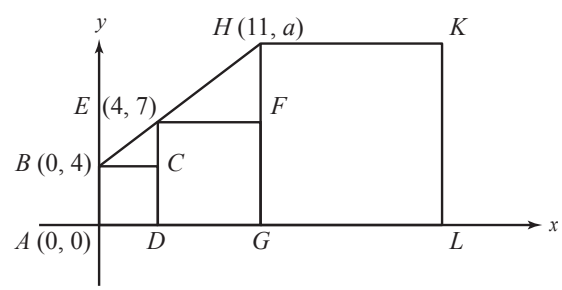
Therefore, $\triangle BCE \sim \triangle EFH$, by Angle−Angle−Angle Triangle Similarity.

Since $\triangle BCE \sim \triangle EFH$, corresponding side lengths are in the same ratio. In particular,

$$\begin{aligned}
\frac{EC}{BC} &= \frac{HF}{EF} \\
\frac{DE - DC}{BC} &= \frac{GH - GF}{EF} \\
\frac{7-4}{4} &= \frac{a-7}{7} \\
\frac{3}{4} &= \frac{a-7}{7} \\
4(a-7) &= 3(7) \\
4a - 28 &= 21 \\
4a &= 49
\end{aligned}$$

$\therefore a = \dfrac{49}{4}$ and the side length of the larger square is $\dfrac{49}{4}$ units.

Solution 3



We proceed by placing the diagram on the xy-plane with A at $(0,0)$ and AL along the x-axis.
The coordinates of B are $(0,4)$, the coordinates of E are $(4,7)$, and the coordinates of H are $(11,a)$.
Let's determine the equation of the line through B, E, H.
Since this line passes through $(0,4)$, it has y-intercept 4.
Since the line passes through $(0,4)$ and $(4,7)$, it has slope $= \frac{7-4}{4-0} = \frac{3}{4}$.
Therefore, the equation of the line is $y = \frac{3}{4}x + 4$.

Since $H(11,a)$ lies on this line, substituting $x = 11$ and $y = a$ into $y = \frac{3}{4}x + 4$, we have

$$a = \frac{3}{4}(11) + 4 = \frac{33}{4} + 4 = \frac{33+16}{4} = \frac{49}{4}$$

$\therefore$ the side length of the larger square is $\dfrac{49}{4}$ units.

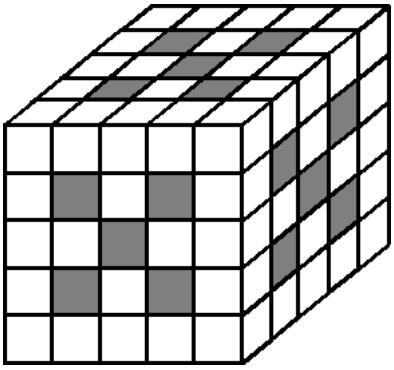
Problème de la semaine

Problème D

Jeu de cube

Un cube mesurant 5 par 5 par 5 est formé de petits cubes identiques mesurant 1 par 1 par 1. Un certain nombre des petits cubes sont enlevés quand on retire quinze colonnes joignant chaque face du cube à celle qui lui est opposée. Les colonnes ainsi enlevées sont en gris dans le diagramme.

Quel est le pourcentage du volume obtenu après avoir enlevé ces quinze colonnes, par rapport à celui du cube original?



Problem of the Week

Problem D and Solution

Cubicles

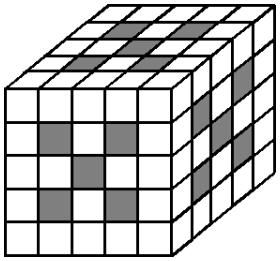
Problem

A 5 by 5 by 5 cube is formed using identical 1 by 1 by 1 cubes. A number of the smaller cubes are removed by punching out the fifteen designated columns from front to back, top to bottom, and side to side. The columns to be removed are shaded grey on the diagram. What percentage of the volume of the original cube remains following the removal of the fifteen columns?

Solution

Solution 1

In this solution, we analyze how many cubes are removed at each of the following stages: when removing the columns from front to back, when removing the columns from top to bottom, and finally when removing the columns from side to side.



When removing columns from the front to the back, 5 smaller cubes are removed from each layer. A total of 25 cubes are removed during this stage.

When removing cubes from top to bottom, the number of cubes removed from each layer is no longer the same. In order from top to bottom, the number of cubes removed at each layer is 5, 1, 4, 1, and 5. A total of 16 additional cubes are removed during this stage.

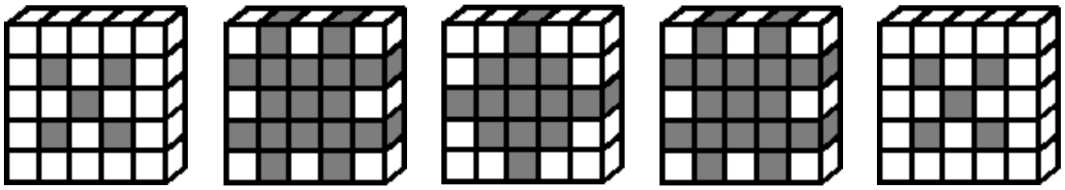
Finally, when removing cubes from side to side, the number of cubes removed at each layer is 5, 1, 4, 1, and 5, the same as the number removed in going from top to bottom. A total of 16 additional cubes are removed during this final stage.

The total number of cubes removed is $25 + 16 + 16 = 57$. The original 5 by 5 by 5 cube had $5 \times 5 \times 5 = 125$ of the smaller 1 by 1 by 1 cubes. The number of cubes remaining is $125 - 57 = 68$. The percentage of the original cube remaining after the removal of the fifteen columns is $68 \div 125 \times 100\% = 54.4\%$.

Solution 1 forces the solver to visualize the solution. The second solution will be more concrete.

Solution 2

After the columns have been removed, peel off each of the layers from front to back. Each layer is illustrated in the diagrams shown below.



In the first layer, 20 of the 1 by 1 by 1 cubes remain. In the second layer, 8 of the 1 by 1 by 1 cubes remain. In the third layer, 12 of the 1 by 1 by 1 cubes remain. In the fourth layer, 8 of the 1 by 1 by 1 cubes remain. And in the final layer, 20 of the 1 by 1 by 1 cubes remain.

A total of $20 + 8 + 12 + 8 + 20 = 68$ of the 1 by 1 by 1 cubes remain. There were 125 of the 1 by 1 by 1 cubes in the original 5 by 5 by 5 cube.

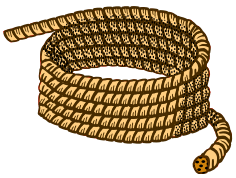
The percentage of the original cube remaining after the removal of the fifteen columns is $68 \div 125 \times 100\% = 54.4\%$.

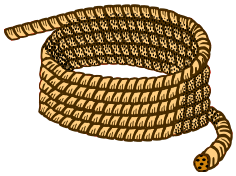
Problème de la semaine

Problème D

Encordé

Une corde de 200 cm de longueur est coupée en quatre morceaux. Trois morceaux sont utilisés pour créer des triangles équilatéraux identiques dont la longueur des côtés est un nombre entier. Le quatrième morceau est utilisé pour former un carré dont la longueur des côtés est aussi un nombre entier. Détermine tous les triangles et le carré possibles en donnant explicitement toutes les longueurs de leurs côtés.





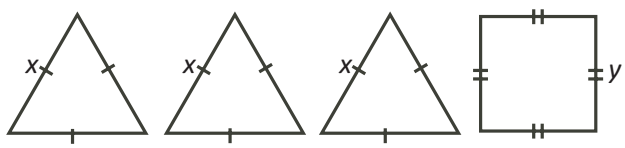
Problem of the Week
Problem D and Solution
Roped In

Problem

A rope of length 200 cm is cut into four pieces. Three of the pieces are used to form identical equilateral triangles with integer side lengths. The fourth piece is used to form a square with integer side lengths. Determine all possible side lengths for each triangle and square.

Solution

Let x represent the integer side length of each equilateral triangle and let y represent the integer side length of the square.



The perimeter of each figure is the length of the piece of rope used to form it. For each triangle, the length of rope is $3x$ and for the square the length of rope is $4y$. The total rope used is $3(3x) + 4y = 9x + 4y$. But the length of the rope is 200 cm. Therefore,

$$\begin{aligned} 9x + 4y &= 200 \\ 9x &= 200 - 4y \\ x &= \frac{4(50-y)}{9} \end{aligned}$$

Since both x and y are integers, $4(50-y)$ must be a multiple of 9. But 4 is not divisible by 9, so $50 - y$ must be divisible by 9. There are five multiples of 9 between 0 and 50, namely 9, 18, 27, 36, and 45. So $50 - y = \{9, 18, 27, 36, 45\}$ and it follows that $y = \{41, 32, 23, 14, 5\}$. The corresponding values of x are computed in the chart below.

y	$4y$	$200-4y$	$x = \dfrac{200-4y}{9}$
41	164	36	4
32	128	72	8
23	92	108	12
14	56	144	16
5	20	180	20

When the side length of the square is 41 cm, the side length of each triangle is 4 cm; when the side length of the square is 32 cm, the side length of each triangle is 8 cm; when the side length of the square is 23 cm, the side length of each triangle is 12 cm; when the side length of the square is 14 cm, the side length of each triangle is 16 cm; and when the side length of the square is 5 cm, the side length of each triangle is 20 cm.

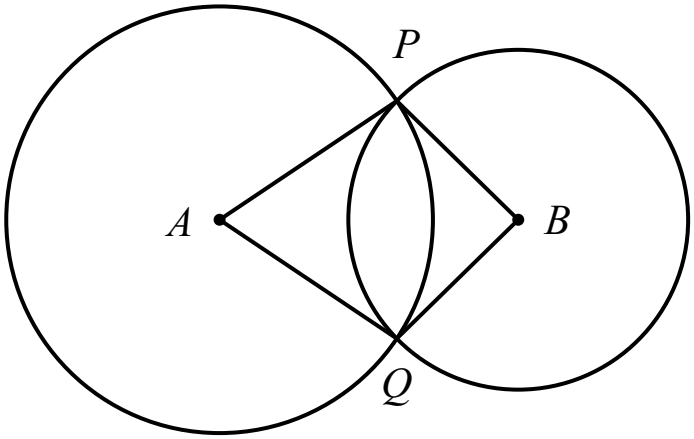
Problème de la semaine

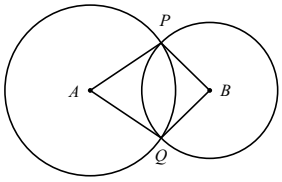
Problème D

Quelle est la relation?

Deux cercles dont les centres sont, respectivement, les points A et B, se croisent aux points P et Q de sorte que $\angle PAQ = 60^\circ$ et $\angle PBQ = 90^\circ$.

Quelle est la relation entre l'aire du cercle ayant pour centre le point A et l'aire de celui dont le centre est le point B?





Problem of the Week

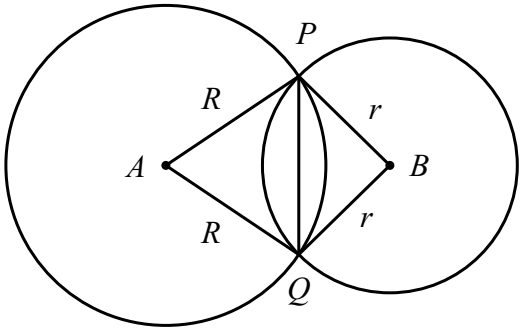
Problem D and Solution

How are We Related?

Problem

Two circles, one with centre A and one with centre B, intersect at points P and Q such that $\angle PAQ = 60^\circ$ and $\angle PBQ = 90^\circ$. How is the area of the circle with centre A related to the area of the circle with centre B?

Solution



Let R be the radius of the circle with centre A and r be the radius of the circle with centre B.

Join P to Q.

We will determine the length of PQ in terms of R and then in terms of r in order to find a relationship between R and r.

Consider $\triangle APQ$. Since $AP = AQ = R$, $\triangle APQ$ is isosceles and so $\angle APQ = \angle AQP$. Since $\angle PAQ = 60^\circ$, $\angle APQ = \angle AQP = \frac{180^\circ - 60^\circ}{2} = 60^\circ$. Therefore, $\triangle APQ$ is equilateral and $PQ = AP = AQ = R$.

Consider $\triangle BPQ$. We are given that $\angle PBQ = 90^\circ$. Therefore, $\triangle BPQ$ is a right-angled triangle. The Pythagorean theorem tells us that $PQ^2 = BP^2 + BQ^2 = r^2 + r^2 = 2r^2$.

We have $PQ = R$ and $PQ^2 = 2r^2$. Therefore, $R^2 = 2r^2$.

The area of the circle with centre B and radius r is πr^2.

The area of the circle with centre A and radius R is

$$\pi R^2 = \pi(2r^2) = 2(\pi r^2) = 2 \times (\text{ the area of the circle with centre } B).$$

Therefore, the area of the circle with centre A is twice the area of the circle with centre B.

Problème de la semaine

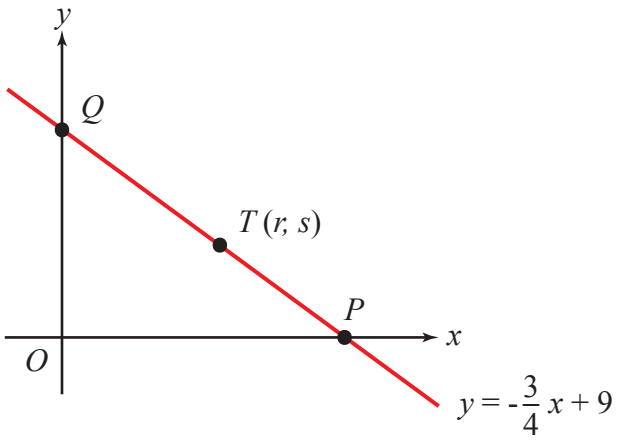
Problème D

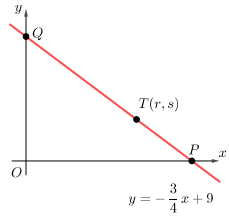
Le point de division

La droite $y = -\frac{3}{4}x + 9$ croise l'axe des x au point P et l'axe des y au point Q.

Le point $T(r, s)$ se situe sur le segment PQ de façon à ce que l'aire de POQ soit trois fois celle de TOP.

Détermine les valeurs de r et s, les coordonnées du point T.





Problem of the Week

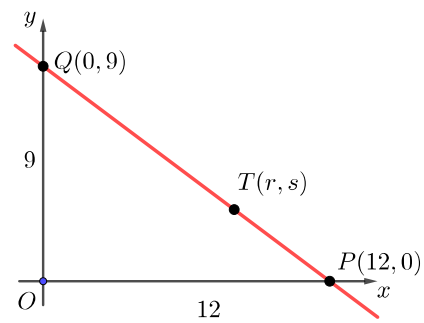
Problem D and Solution

A Point of Division

Problem

The line $y = -\frac{3}{4}x + 9$ crosses the x-axis at P and the y-axis at Q. Point $T(r, s)$ lies on the line segment PQ such that the area of $\triangle POQ$ is three times the area of $\triangle TOP$. Determine the values of r and s, the coordinates of T.

Solution

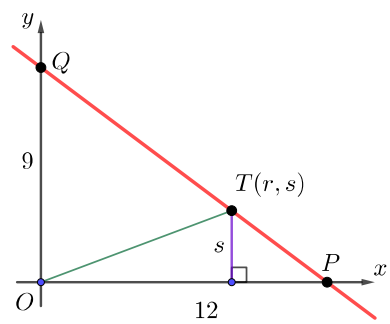


We begin by calculating the coordinates of P and Q, the x- and y-intercepts of the line $y = -\frac{3}{4}x + 9$.

Since the equation of the line is written in the form $y = mx + b$ where b is the y-intercept of the line, the y-intercept is 9 and so the coordinates of Q are $(0, 9)$. To determine the x-intercept, set $y = 0$ to obtain $0 = -\frac{3}{4}x + 9 \implies \frac{3}{4}x = 9 \implies x = 12$. Thus, P has coordinates $(12, 0)$.

We now present two different solutions to the problem.

Solution 1



Since $\triangle POQ$ is a right triangle with base $PO = 12$ and height $OQ = 9$, using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have $\text{area}(\triangle POQ) = \frac{12 \times 9}{2} = 54$.

Since the area of $\triangle POQ$ is three times the area of $\triangle TOP$, $\text{area}(\triangle TOP) = \frac{1}{3}(\text{area}(\triangle POQ)) = \frac{1}{3}(54) = 18$.

$\triangle TOP$ has area 18, base $PO = 12$ and height s. Using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have

$$\begin{aligned} \text{area}(\triangle TOP) &= \frac{PO \times s}{2} \\ 18 &= \frac{12 \times s}{2} \\ 18 &= 6s \\ \therefore\ s &= 3 \end{aligned}$$

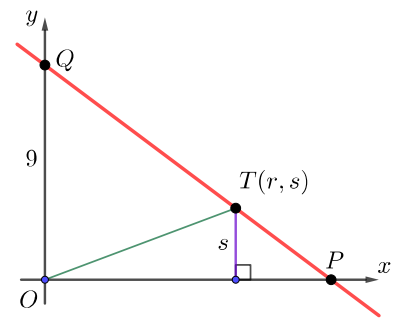
$T(r, s)$ lies on the line $y = -\frac{3}{4}x + 9$ and $s = 3$ so we can substitue $x = r$ and $y = 3$

$$\begin{aligned} 3 &= -\frac{3}{4}r + 9 \\ \frac{3}{4}r &= 6 \\ \therefore\ r &= 8 \end{aligned}$$

Therefore, T is the point $(r, s) = (8, 3)$.

Solution 2

If two triangles have equal bases, the areas of the triangles are proportional to the heights of the triangles.



$\triangle POQ$ and $\triangle TOP$ have the same base, OP.

Since the area of $\triangle POQ$ is 3 times the area of $\triangle TOP$, then the height of $\triangle POQ$ is 3 times the height of $\triangle TOP$. In other words, the height of $\triangle TOP$ is $\frac{1}{3}$ the height of $\triangle POQ$. $\triangle POQ$ has height $OQ = 9$ and $\triangle TOP$ has height s. Therefore, $s = \frac{1}{3}(OQ) = \frac{1}{3}(9) = 3$.

Since $T(r, s)$ lies on the line $y = -\frac{3}{4}x + 9$, we have

$$\begin{aligned} s &= -\frac{3}{4}r + 9 \\ 3 &= -\frac{3}{4}r + 9 \\ \frac{3}{4}r &= 6 \\ \therefore\ r &= 8 \end{aligned}$$

Therefore, T is the point $(r, s) = (8, 3)$.

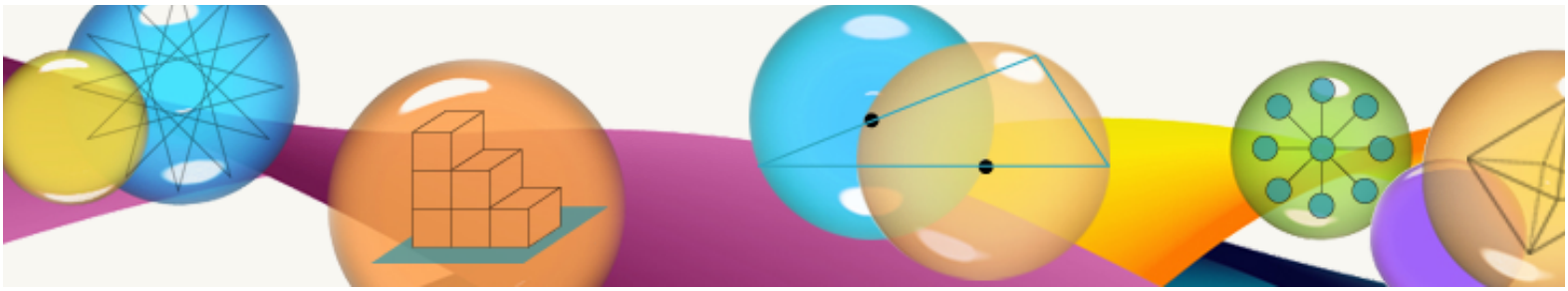
Note that it was actually unnecessary to find the x-intercept for the second solution as it was never used in the second solution.

For Further Thought:

Find the coordinates of S, another point on line segment QP, so that

$$\text{the area of } \triangle SOQ = \text{ the area of } \triangle TOP,$$

thus creating three triangles of equal area. How are the points Q, S, T, and P related?



Sens du nombre et algèbre

Amène-moi à la couverture

Problème de la semaine

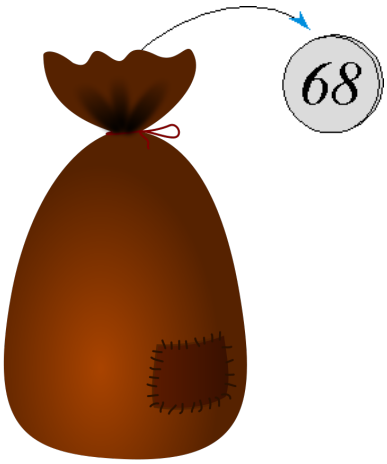
Problème D

Des jetons jetés

Des jetons sont placés dans un sac. Un nombre entier positif est inscrit sur l'un des côtés de chaque jeton. Il est possible qu'il existe plus d'un jeton ayant le même nombre. La moyenne de tous les nombres est de 56.

Si l'on retire un jeton sur lequel est inscrit le numéro 68, la moyenne devient maintenant 55.

Détermine le plus grand nombre entier possible qui pourrait paraître sur un des jetons.



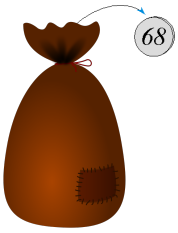
Problem of the Week

Problem D and Solution

Taken Token

Problem

Some tokens are placed in a bag. Each of the tokens has a positive integer stamped on one of its sides. It is possible that more than one token in the bag has the same number stamped on it. The average of all of the numbers stamped on the tokens in the bag is 56. If a token with the number 68 on it is removed from the bag, the average of the numbers stamped on the remaining tokens is 55. Determine the largest possible integer that could appear on one of the tokens in the bag.



Solution

To calculate an average, determine the sum of the numbers in the set and divide by the number of numbers in the set. It follows that the sum of the numbers in a set is the average times the number of numbers in the set.

Let n represent the number of tokens in the bag.

The total of the numbers on all of the tokens in the bag is $56n$. After the token with the number 68 stamped on it is removed, the total of the numbers on the remaining tokens is $(56n - 68)$ and there are $(n - 1)$ tokens remaining in the bag. The average of the numbers stamped on the tokens remaining in the bag is 55 so it follows that

$$\begin{aligned} \frac{56n - 68}{n - 1} &= 55 \\ 56n - 68 &= 55(n - 1) \\ 56n - 68 &= 55n - 55 \\ n &= 13 \end{aligned}$$

Since $n = 13$, there were originally 13 tokens in the bag and the total of the numbers stamped on the tokens in the bag was $56n = 56(13) = 728$.

To determine the highest possible value on a token, stamp eleven of the tokens with the smallest possible positive integer, namely 1. The twelfth token has the number 68 stamped on it. The largest possible value that could be stamped on the remaining token is, $728 - 11 \times 1 - 68 = 649$.

As an extension, consider how the answer would change if no two tokens were stamped with the same number.

Problème de la semaine

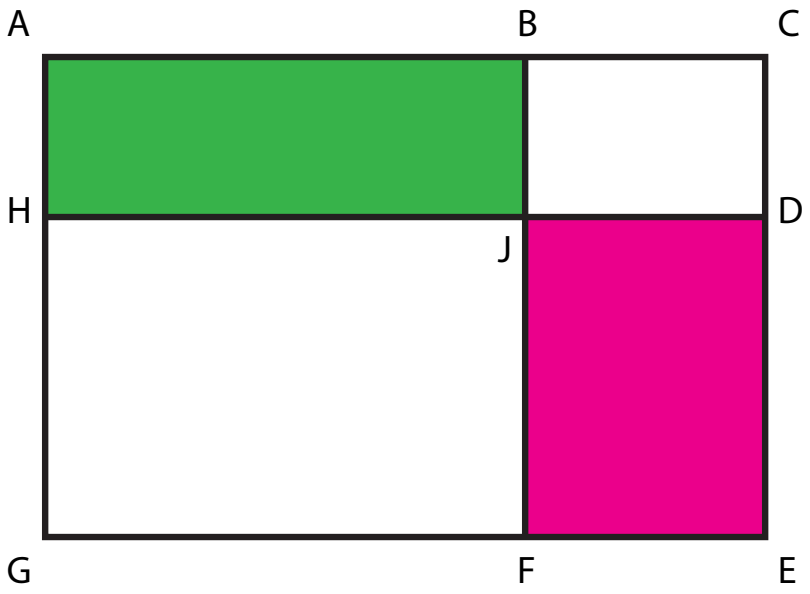
Problème D

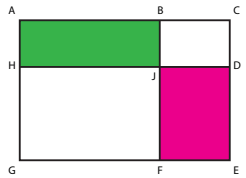
Maximiser l'aire

Les rectangles $ABJH$ et $JDEF$, dont la longueur de chacun de leurs côtés est un nombre entier, se touchent au coin J de façon à ce que le segment de droite HJD soit perpendiculaire au segment de droite BJF. Les deux rectangles sont inclus dans un plus grand rectangle $ACEG$, comme indiqué dans le diagramme ci-dessous.

L'aire du rectangle $ABJH$ est de 6 cm^2 et l'aire du rectangle $JDEF$ est de 15 cm^2.

Détermine la plus grande aire possible du rectangle $ACEG$. Il est à noter que le diagramme n'est pas à l'échelle.





Problem of the Week

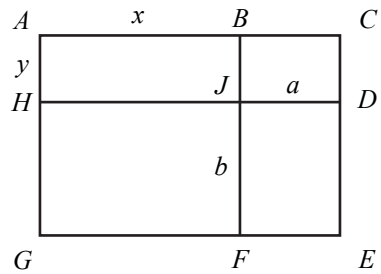
Problem D and Solution

Maximize the Area

Problem

Two rectangles, $ABJH$ and $JDEF$, with integer side lengths, share a common corner at J such that HJD and BJF are perpendicular line segments. The two rectangles are enclosed by a larger rectangle $ACEG$, as shown. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2. Determine the largest possible area of the rectangle $ACEG$. Note that the diagram is not intended to be to scale.

Solution



Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$.

Therefore,
$$\begin{aligned} AB &= HJ = GF = x, \\ AH &= BJ = CD = y, \\ BC &= JD = FE = a, \text{ and} \\ HG &= JF = DE = b. \end{aligned}$$

$$\begin{aligned} \text{Then area}(ACEG) &= \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG) \\ &= 6 + ya + 15 + xb \\ &= 21 + ya + xb \end{aligned}$$

Since the area of rectangle $ABJH$ is 6 cm^2 and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

Since the area of rectangle $JDEF$ is 15 cm^2 and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

To maximize the area, we need to pick the values of x, y, a, b which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of a and b. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of a and b. (As an extension, we will leave it to you to think about why this is the case.)

Case 1: $x = 1$ cm, $y = 6$ cm and $a = 1$ cm, $b = 15$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42 \text{ cm}^2$$

Case 2: $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112 \text{ cm}^2$$

Case 3: $x = 1$ cm, $y = 6$ cm and $a = 3$ cm, $b = 5$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44 \text{ cm}^2$$

Case 4: $x = 1$ cm, $y = 6$ cm and $a = 5$ cm, $b = 3$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54 \text{ cm}^2$$

Case 5: $x = 2$ cm, $y = 3$ cm and $a = 1$, $b = 15$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54 \text{ cm}^2$$

Case 6: $x = 2$ cm, $y = 3$ cm and $a = 15$, $b = 1$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68 \text{ cm}^2$$

Case 7: $x = 2$ cm, $y = 3$ cm and $a = 3$, $b = 5$ cm

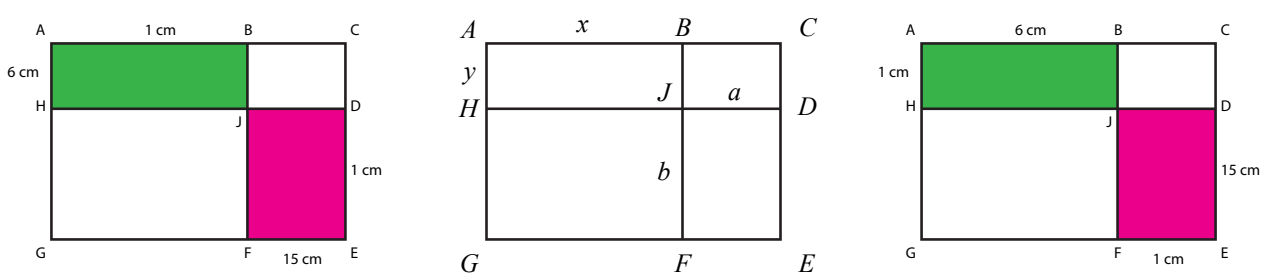
$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40 \text{ cm}^2$$

Case 8: $x = 2$ cm, $y = 3$ cm and $a = 5$, $b = 3$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42 \text{ cm}^2$$

We see that the maximum area is 112 cm^2, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm^2.



Problème de la semaine

Problème D

Ne pas perdre une vente!

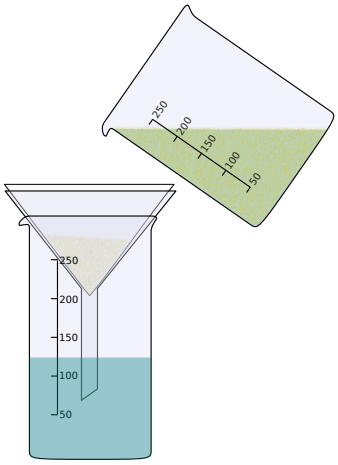
Un client entre dans une boutique automobile à la recherche d'antigel contenant 8% de glycérine. C'est la concentration recommandée dans le manuel d'utilisation de son véhicule.

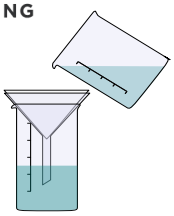
La propriétaire du magasin possède:

- une bouteille d'antigel de trois litres contenant 7% de glycérine,
- une bouteille de quatre litres contenant 20% de glycérine, et
- une réserve d'antigel contenant 5% de glycérine.

Pour ne pas rater une opportunité de vendre, la propriétaire estime qu'elle peut créer un antigel contenant 8% de glycérine, si elle combine tout le contenu de la bouteille de trois litres avec tout le contenu de la bouteille de quatre litres, et ensuite si elle y ajoute une quantité nécessaire d'antigel provenant de sa réserve.

Combien de litres d'antigel contenant 5% de glycérine, la propriétaire doit-elle ajouter pour créer une solution d'antigel contenant 8% de glycérine? (Tu peux supposer que la propriétaire possède suffisamment d'antigel à 5% de glycérine pour créer la composition d'antigel désirée.)





Problem of the Week

Problem D and Solution

Don't Want to Miss a Sale

Problem

A customer came into an auto shop requiring a quantity of anti-freeze of which 8% is glycerine. This was the percentage recommended in her owner's manual. The shop owner has the following:

- a three litre bottle of anti-freeze of which 7% is glycerine,
- a four litre bottle of anti-freeze of which 20% is glycerine, and
- a bulk supply of antifreeze of which 5% is glycerine.

The owner, not wanting to miss a sale, felt that she could create an anti-freeze mixture with the required percentage by combining the contents of the entire three litre bottle with the contents of the entire four litre bottle and then adding some anti-freeze from the bulk supply. How many litres of anti-freeze from the bulk supply must be added to create a quantity of anti-freeze of which 8% is glycerine? (You may assume that the shop owner has enough of the bulk supply to create the appropriate mixture.)

Solution

Let x be the number of litres from the bulk supply..

The 3 L container of anti-freeze of which 7% is glycerine has $0.07 \times 3 = 0.21$ L of glycerine.

The 4 L container of anti-freeze of which 20% is glycerine has $0.20 \times 4 = 0.80$ L of glycerine.

In the x L from the bulk supply there is $0.05 \times x = \ 0.05x$ L of glycerine.

Therefore, the total amount of glycerine in the final mixture is
$0.21 + 0.80 + 0.05x = (1.01 + 0.05x)$ L.

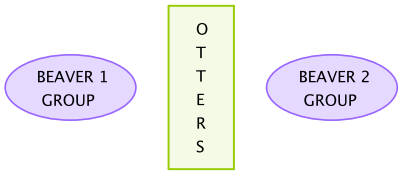
The final mixture contains $3 + 4 + x = (7 + x)$ L of liquid, of which 8% is glycerine.
Therefore, $0.08 \times (7 + x) = (0.56 + 0.08x)$ L of the final mixture is glycerine.

Since we have shown that the amount of glycerine in the final mixture is $(1.01 + 0.05x)$ L and $(0.56 + 0.08x)$ L, we must have

$$\begin{aligned} 1.01 + 0.05x &= 0.56 + 0.08x \\ 0.05x - 0.08x &= 0.56 - 1.01 \\ -0.03x &= -0.45 \\ x &= 15 \end{aligned}$$

Therefore, 15 L of the bulk anti-freeze is required.

Problème de la semaine
Problème D
Tu dis quoi?

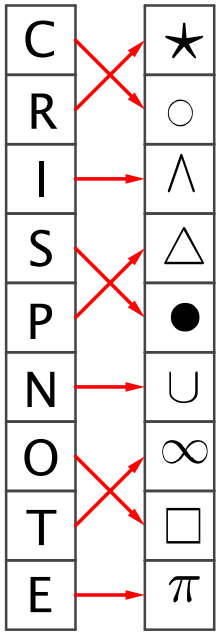


Les Castors jouent un jeu contre les Loutres. Le groupe 1 des Castors doit envoyer un message secret au groupe 2 des Castors, cependant le message doit passer par une zône occupée par les Loutres.

Les Castors décident d'utiliser un mécanisme, nommé la machine Énigme-B, pour déguiser les messages qu'ils s'enverront entre eux. La machine possède deux rotors. Le rotor de gauche se déplace lorsqu'une lettre est tapée, tel que décrit ci-dessous. Le rotor de droite est immobile.

La machine Énigme-B fonctionne comme suit :

- La machine commence dans la position de DÉPART, indiqueée par le diagramme de droite.
- Une lettre sur le rotor de gauche est codée au symbole correspondant du rotor de droite. Si, par exemple, la lettre P est tapée à partir de la position de DÉPART, elle sera d'abord codée au symbole $\triangle$. Après avoir tapé la première lettre, le rotor de gauche monte d'une position. La premiére lettre devient la derniére lettre.
- Une deuxième lettre est tapée et codée au symbole correspondant sur le rotor de droite. Si, par exemple, la lettre O est tapée, elle sera codée au symbole $\cup$. Après avoir tapé la deuxième lettre, le rotor de gauche se déplace de deux positions vers le haut. Les deux premières lettres deviennent maintenant les deux dernières, dans le même ordre.
- Une troisième lettre est tapée et codée au symbole correspondant sur le rotor de droite. Si, par exemple, la lettre tapée est R, elle sera codée au symbole ∞. Après avoir tapé la troisième lettre, le rotor de gauche monte de trois positions et les trois premières lettres deviennent les trois dernières lettres, dans le même ordre.
- Une quatrième lettre est tapée et codée au symbole correspondant sur le rotor de droite. Si, par exemple, la lettre tapée est T, elle sera codée au symbole $\star$. Après avoir tapé la quatrième lettre, le rotor de gauche montre de quatre positions. Les quatre premières lettres deviennent les quatre dernières lettres, dans le même ordre.



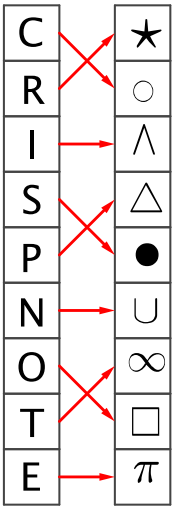
Le processus continue jusqu'à ce que le bouton ENVOYER (qui ne figure pas ici) soit appuyé. Une fois le message envoyé, le rotor de gauche retourne à la position de DÉPART.

Un message à quatre lettres, "PORT", est ainsi crypté par les symboles $\triangle \cup \infty \star$. Le processus d'encodage est illustré à la page suivante.

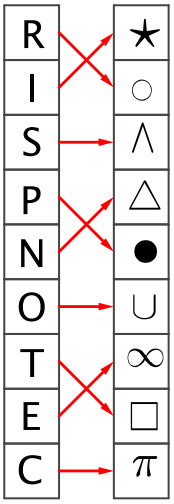
Le groupe 1 des Castors envoie le message "SCRIPT TOP SECRET" et appui sur le bouton ENVOYER. En supposant que le rotor de gauche est à la position de DÉPART et que les espaces entre les mots sont ignorés, quel sera le message codé reçu par le groupe 2 des Castors?

Illustration par les diagrammes suivants de l'encodage du mot "PORT" .

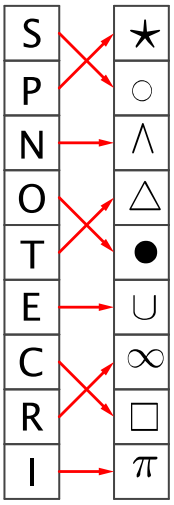
DÉPART ⇒ Le rotor de gauche monte d'une position. La première lettre devient la dernière. ⇒ Le rotor de gauche monte de 2 positions. Les 2 premières lettres deviennent les 2 dernières.



P → △

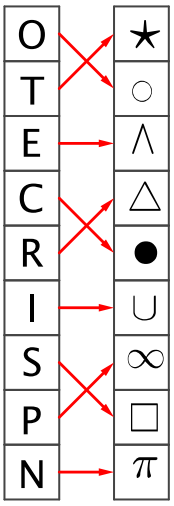


O → ∪

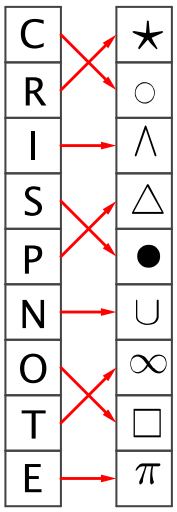


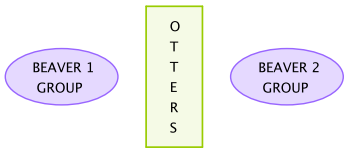
R → ∞

⇒ Le rotor de gauche monte de 3 positions. Les 3 premières lettres deviennent les 3 dernières. ⇒ ENVOYER Retour à la position de DÉPART.

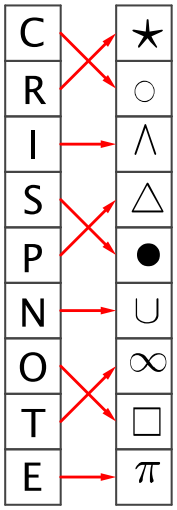


T → ★





Problem of the Week
Problem D and Solution
Say What?



Problem

The Beavers are playing a game with the Otters. The Beaver 1 group needs to communicate secretly with the Beaver 2 group, but their message will pass through a zone controlled by the Otters. The Beavers decide to use a mechanism called the B-Enigma machine to encrypt (disguise) their messages while sending them from one side to the other. The device has two rotors. As described below, the left rotor moves after a letter is typed. The right rotor never moves.

The following is a description of how the B-Enigma machine works.

The machine begins in the START position illustrated in the diagram up to the right. A letter on the left rotor is encrypted to the corresponding symbol on the right rotor. If, for example, P is the letter typed first from the START position, it will be encrypted as $\triangle$. After typing the first letter, the left rotor will move up one position. The top letter moves down to the bottom. A second letter is typed and encrypted to the symbol on the right rotor. If, for example, O is typed second, it will be encrypted as $\cup$. After typing the second letter, the left rotor will move up two positions. The top two letters will move to the bottom and stay in the same order. A third letter is typed and encrypted to the symbol on the right rotor. If, for example, R is typed third, it will be encrypted as ∞. After typing the third letter, the left rotor will move up three positions. The top three letters will move to the bottom and stay in the same order. A fourth letter is typed and encrypted to the symbol on the right rotor. If, for example, T is typed fourth, it will be encrypted as $\star$. After typing the fourth letter, the left rotor will move up four positions. The top four letters will move to the bottom and stay in the same order.

The procedure and pattern repeat until the SEND button (not shown) is pressed. After a message is sent, the left rotor automatically returns to the START position. Our four-letter message was PORT and it was encrypted $\triangle \cup \infty \star$.

The Beaver 1 Group sends the message "TOP SECRET SCRIPT" and then presses SEND. Assuming the left rotor is in the START position and spaces in the message are ignored, what is the encrypted message received by the Beaver 2 Group.

Solution

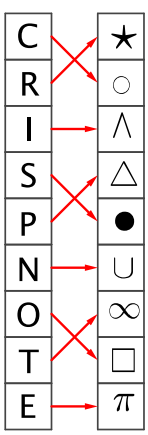
Let the START position of the left rotor be Position 0. If the left rotor is in Position 1, the letters have moved 1 position up from their initial position. If the left rotor is in Position 2, the letters have moved 2 positions up from their initial position. In fact there are actually only 9 positions that the rotor can be in. If the rotor moves 9 positions it will be back in the START position again. If the left rotor has moved 9 positions or more, we can determine its position relative to the START position by subtracting multiples of 9 from the total number of moves until we obtain a position number from 0 to 8.

The table allows us to determine where the left rotor is when a letter is encrypted. For example, when the O is encrypted, the left rotor is in Position 1 and the O is encrypted $\cup$. When the I is encrypted, the left rotor has moved a total of 78 positions. The closest multiple of 9 to 78 is 72 so, relative to the START Position, each letter has moved $78 - 72 = 6$ positions. Using the Position 6 illustration below, the letter I is encrypted $\cup$.

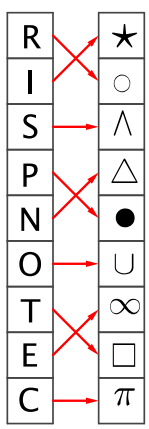
Letter to Encrypt	Number of Positions Moved before encryption	Position Relative to START Position	Letter is encrypted to
T	0	0	∞
O	0+1=1	1	$\cup$
P	1+2=3	3	$\star$
S	3+3=6	6	$\square$
E	6+4=10	10-9=1	∞
C	10+5=15	15-9=6	$\bullet$
R	15+6=21	21-18=3	∞
E	21+7=28	28-27=1	∞
T	28+8=36	36-36=0	∞
S	36+9=45	45-45=0	$\bullet$
C	45+10=55	55-54=1	π
R	55+11=66	66-63=3	∞
I	66+12=78	78-72=6	$\cup$
P	78+13=91	91-90=1	$\bullet$
T	91+14=105	105-99=6	$\star$

It turns out that we only need four positions of the left rotor relative to the START position. We need position 0 (START Position), position 1, position 3, and position 6.

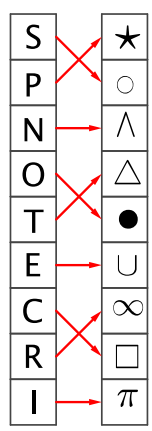
Position 0



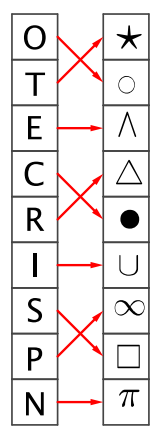
Position 1



Position 3



Position 6



The message TOP SECRET SCRIPT is encrypted $\infty \cup \star \square \infty \bullet \infty \infty \infty \bullet \pi \infty \cup \bullet \star$.

The message SCRIPT TOP SECRET is encrypted $\bullet \pi \infty \cup \bullet \star \triangle \cup \triangle \bullet \infty \square \triangle \infty \star$.

See the next page for further discussion and comments about this problem.

For Further Thought

- How would the Beaver 2 group decrypt (decipher) the message?
- When the rotor moves $n + 9m$ positions from the start position, $0 \leq n \leq 8, n \in I$, $m \geq 1, m \in I$, its final position will be n positions from the start position. A study of modular arithmetic could help if you wish to explore this further.
- In our example, there were 9 letters on the left rotor and 9 symbols on the right rotor. The cycle of positions used on the left rotor only caused positions 0, 1, 3, and 6 of the left rotor to be used. Is there a size of rotor so that every position of the rotor would be used in the encryption process? Experiment with a few different sizes.

Problème de la semaine

Problème D

PRODUITivité

Le *produit des chiffres* d'un nombre entier positif est le produit de tous les chiffres qui composent ce nombre.

Par exemple, le produit des chiffres du nombre 234 est $2 \times 3 \times 4 = 24$. Il existe d'autres nombres dont le produit des chiffres est aussi 24. Par exemple, 2233, 113181 et 38, ont tous un produit des chiffres égal à 24. Le nombre 38 est le plus petit nombre entier positif dont le produit des chiffres est 24.

Il existe plusieurs nombres entiers positifs dont le produit des chiffres est 2000.

Détermine le plus petit nombre entier positif dont le produit des chiffres de 2000.

Problem of the Week

Problem D and Solution

PRODUCTivity

Problem

The *"digit product"* of a positive integer is the product of the individual digits of the integer. For example, the digit product of 234 is $2 \times 3 \times 4 = 24$. Other numbers also have a digit product of 24. For example, 2223, 113 181 and 38 each have a digit product of 24. The number 38 is the smallest positive integer with a digit product of 24. There are many positive integers whose digit product is 2000. Determine the smallest positive integer whose digit product is 2000.

Solution

Let N be the smallest positive integer whose digit product is 2000.

In order to find N, we must find the minimum possible number of digits whose product is 2000. This is because if the integer a has more digits than the integer b, then $a > b$. Once we have determined the digits that form N, then the integer N is formed by writing those digits in increasing order.

Note that the digits of N cannot include 0, or else the digit product of N would be 0. Also, the digits of N cannot include 1, otherwise we could remove the 1 and obtain an integer with fewer digits (and thus, a smaller integer) with the same digit product. Therefore, the digits of N will be between 2 and 9, inclusive.

Since the digit product of N is 2000, we will use the prime factorization of 2000 to help determine the digits of N:

$$2000 = 2^4 \times 5^3$$

In order for a digit to have a factor of 5, the digit must equal 5. Therefore, three of the digits of N are 5.

The remaining digits of N must have a product of $2^4 = 16$. We need to find a combination of the smallest number of digits whose product is 16. We cannot have one digit whose product is 16, but we can have two digits whose product is 16. In particular, $16 = 2 \times 8$ and $16 = 4 \times 4$.

Therefore, N has 5 digits. They are 5, 5, 5, 2, 8 or 5, 5, 5, 4, 4. In order for N to be as small as possible, its digits must be in increasing order. The smallest possible number formed by the digits 5, 5, 5, 2, 8 is 25 558. The smallest possible number formed by the digits 5, 5, 5, 4, 4 is 44 555.

Since $25\,558 < 44\,555$, the smallest N is 25 558. That is, the smallest positive integer with a digit product of 2000 is 25 558.

Problème de la semaine

Problème D

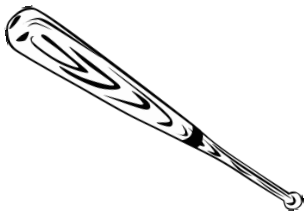
Le classement

Dans une ligue de balle-molle à quatre équipes, chaque équipe joue 4 fois contre chacune des autres équipes.

Les équipes gagnent 3 points pour une victoire, 1 point pour un match nul et aucun point pour une défaite.

Les points accumulés à la fin de la saison sont :

Les Lions	22
Les Tigres	19
Les Gendarmes	14
Les Nobles	12



Dans combien de matchs y a-t-il eu une victoire et combien de matchs se sont terminés à égalité?

Problem of the Week

Problem D and Solution

The Standings

Problem

In a four team softball league, each team has played every other team 4 times. A team earned 3 points for a win, 1 point for a tie and no points for a loss. The total accumulated points were: Lions 22, Tigers 19, Mounties 14, and Royals 12. How many games ended in a win and how many games ended in a tie?

Solution

We begin by calculating the total number of games played. Since each team played every other team 4 times, each team played $3 \times 4 = 12$ games. Since there are four teams, a total of $\frac{4 \times 12}{2} = 24$ games were played. We divide by 2 since each game is counted twice. For example, the Lions playing the Tigers is the same as the Tigers playing the Lions.

In games where one team won and one team lost, one team earned 3 points and the other 0 points, so a total of 3 points were awarded. In games that resulted in a tie, both teams earned 1 point, so a total of 2 points were awarded.

If there were 0 ties, then 24 games would result in $24 \times 3 = 72$ points being awarded. However, $22 + 19 + 14 + 12 = 67$ points were actually awarded in all of the games. Since a total of 3 points were awarded when there was a win and a total of 2 points were awarded when there was a tie, every point below 72 must represent a tie. Since $72 - 67 = 5$, there must have been 5 ties. Since 24 games were played, $24 - 5 = 19$ games resulted in a win.

Therefore, there were 19 games that ended in a win and 5 games ended in a tie.

We should check that there is a combination of wins, ties and losses that satisfies the conditions in the problem. Indeed, one possibility is:

Team Name	Wins	Ties	Losses	Total Points
Lions	7	1	4	22
Tigers	6	1	5	19
Mounties	3	5	4	14
Royals	3	3	6	12
TOTALS	19	10	19	67

Notice, in the chart, that there are a total of 10 ties. That means that 5 games ended in a tie and a total of 10 points were awarded for ties.

Extension: There are 5 other combinations of wins, ties and losses that satisfy the conditions of the problem. Can you find them all?

Problème de la semaine

Problème D

Répétition par produit

Un nombre entier positif doit être incrit dans chacune des cases ci-dessous.

Les nombres peuvent être répétés, mais le produit de quatre nombres contigus doit toujours être égal à 120.

Détermine toutes les valeurs possibles de x.

		2			4			x			3		

Problem of the Week

Problem D and Solution

Repetition By Product

Problem

		2			4			x			3		

A positive integer is to be placed in each box. Integers may be repeated, but the product of any four adjacent integers is always 120. Determine all possible values for x.

Solution

In both solutions, let a_1 be the integer placed in the first box, a_2 the integer placed in the second box, a_4 the integer placed in the fourth box, and so on, as shown below.

a_1	a_2	2	a_4	a_5	4	a_7	a_8	x	a_{10}	a_{11}	3	a_{13}	a_{14}

Solution 1

Consider boxes 3 to 6. Since the product of any four adjacent integers is 120, we have $2 \times a_4 \times a_5 \times 4 = 120$. Therefore, $a_4 \times a_5 = \frac{120}{2\times 4} = 15$. Since a_4 and a_5 are positive integers, there are 4 possibilities: $a_4 = 1$ and $a_5 = 15$, or $a_4 = 15$ and $a_5 = 1$, or $a_4 = 3$ and $a_5 = 5$, or $a_4 = 5$ and $a_5 = 3$.

In each of the four cases, we will have $a_7 = 2$. We can see why by considering boxes $4 - 7$. We have $a_4 \times a_5 \times 4 \times a_7 = 120$, or $15 \times 4 \times a_7 = 120$, since $a_4 \times a_5 = 15$. Therefore, $a_7 = \frac{120}{15\times 4} = 2$.

<u>Case 1:</u> $a_4 = 1$ and $a_5 = 15$

Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $15 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{15\times 4\times 2} = 1$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $4 \times 2 \times 1 \times x = 120$, or $x = \frac{120}{4\times 2} = 15$.
Let's check that $x = 15$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 9 to 12. If $x = 15$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{15\times 3} = \frac{8}{3}$. But a_{10} and a_{11} must both be integers, so is not possible for $a_{10} \times a_{11} = \frac{8}{3}$. Therefore, it must not be possible for $a_4 = 1$ and $a_5 = 15$, and so we find that there is no solution for x in this case.

<u>Case 2:</u> $a_4 = 15$ and $a_5 = 1$

Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $1 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{4\times 2} = 15$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times 2\times 15} = 1$.
Let's check that $x = 1$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 7 to 10. Since $a_7 = 2$, $a_8 = 15$ and $x = 1$, then $a_{10} = \frac{120}{2\times 15\times 1} = 4$. Similarly, $a_{11} = \frac{120}{15\times 1\times 4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 1 \times 4 \times 2 \times 3 = 24 \neq 120$. Therefore, it must not be possible for $a_4 = 15$ and $a_5 = 1$. There is no solution for x in this case.

Case 3: $a_4 = 3$ and $a_5 = 5$

Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $5 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{5\times4\times2} = 3$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times3} = 5$.
Let's check that $x = 5$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 7 to 10. Since $a_7 = 2$, $a_8 = 3$ and $x = 5$, then $a_{10} = \frac{120}{2\times3\times5} = 4$. Similarly, $a_{11} = \frac{120}{3\times5\times4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 5 \times 4 \times 2 \times 3 = 120$. Therefore, the condition that $a_{12} = 3$ is satisfied in the case where $a_4 = 3$ and $a_5 = 5$. If we continue to fill out the entries in the boxes, we obtain the entries shown in the diagram below.

5	4	2	3	5	4	2	3	5	4	2	3	5	4

We see that $x = 5$ is a possible solution. However, is it the only solution? We have one final case to check.

Case 4: $a_4 = 5$ and $a_5 = 3$

Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $3 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{3\times4\times2} = 5$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times5} = 3$.
Let's check that $x = 3$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 9 to 12. If $x = 3$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{3\times3} = \frac{40}{3}$. But a_{10} and a_{11} must both be integers, so it is not possible for $a_{10} \times a_{11} = \frac{40}{3}$. Therefore, it must not be possible for $a_4 = 5$ and $a_5 = 3$, and so we find that there is no solution for x in this case.

Therefore, the only possible value for x is $x = 5$.

Solution 2

You may have noticed a pattern for the a_i's in Solution 1. We will explore this pattern.

a_1	a_2	2	a_4	a_5	4	a_7	a_8	x	a_{10}	a_{11}	3	a_{13}	a_{14}

Since the product of any four integers is 120, $a_1a_2a_3a_4 = a_2a_3a_4a_5 = 120$. Since both sides are divisible by $a_2a_3a_4$, and each is a positive integer, then $a_1 = a_5$.
Similarly, $a_2a_3a_4a_5 = a_3a_4a_5a_6 = 120$, and so $a_2 = a_6$.
In general, $a_na_{n+1}a_{n+2}a_{n+3} = a_{n+1}a_{n+2}a_{n+3}a_{n+4}$, and so $a_n = a_{n+4}$.
We can use this along with the given information to fill out the boxes as follows:

x	4	2	3	x	4	2	3	x	4	2	3	x	4

Therefore, $4 \times 2 \times 3 \times x = 120$ and so $x = \frac{120}{4\times2\times3} = 5$.

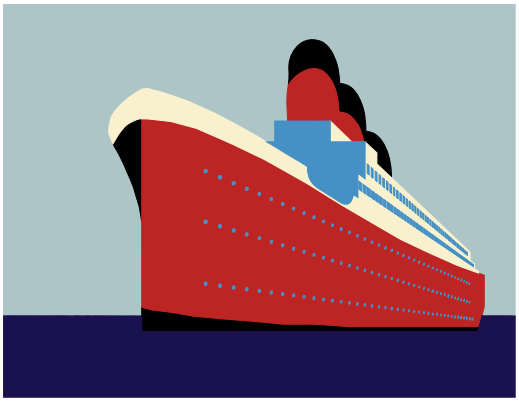
Problème de la semaine

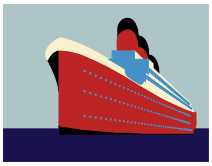
Problème D

Tout le monde à bord!

Un passager en croisière en Alaska a demandé au capitaine combien de passagers étaient à bord du paquebot. Le capitaine a répondu : $\frac{1}{6}$ des passagers sont des personnes âgées, $\frac{1}{4}$ des passagers sont des enfants ou des adolescents, il y a trois fois plus d'adultes que d'adolescents, et il y a 138 enfants à bord.

Combien y a-t-il de passagers sur le paquebot de croisière?





Problem of the Week
Problem D and Solution
All Aboard!

Problem

The captain of an Alaskan cruise was asked by one of the passengers how many guests were on board. The captain replied: $\frac{1}{6}$ of our guests are seniors, $\frac{1}{4}$ of our guests are children or teenagers, there are three times as many adults as teenagers, and there are 138 children on board. How many passengers are on board the cruise ship?

Solution

Let n be the total number of passengers.
Let s be the number of seniors, a be the number of adults, t be the number of teenagers, and c be the number of children.

Therefore, $n = s + a + t + c$.

Since $\frac{1}{6}$ of the passengers are seniors, $s = \frac{1}{6}n$.
Since $\frac{1}{4}$ of the passengers are children or teenagers, $c + t = \frac{1}{4}n$.
It is given that $c = 138$. Therefore, this becomes $138 + t = \frac{1}{4}n$, or $t = \frac{1}{4}n - 138$.
Since there are three times as many adults as teenagers, $a = 3t = 3(\frac{1}{4}n - 138)$.

Substituting into $n = s + a + t + c$, we have

$$\begin{aligned}
n &= \left(\frac{1}{6}n\right) + 3\left(\frac{1}{4}n - 138\right) + \left(\frac{1}{4}n - 138\right) + 138 \\
n &= \frac{1}{6}n + \frac{3}{4}n - 414 + \frac{1}{4}n - 138 + 138 \\
n &= \frac{1}{6}n + n - 414 \\
n &= \frac{7}{6}n - 414
\end{aligned}$$

Rearranging, we have $\frac{7}{6}n - n = 414$, and so $\frac{1}{6}n = 414$ or $n = 2484$.
Therefore, there are 2484 passengers on board the cruise ship.

Although not required, we can determine the number of adults is 1449, the number of seniors is 414, the number of teenagers is 483. We can use this to verify the given information.

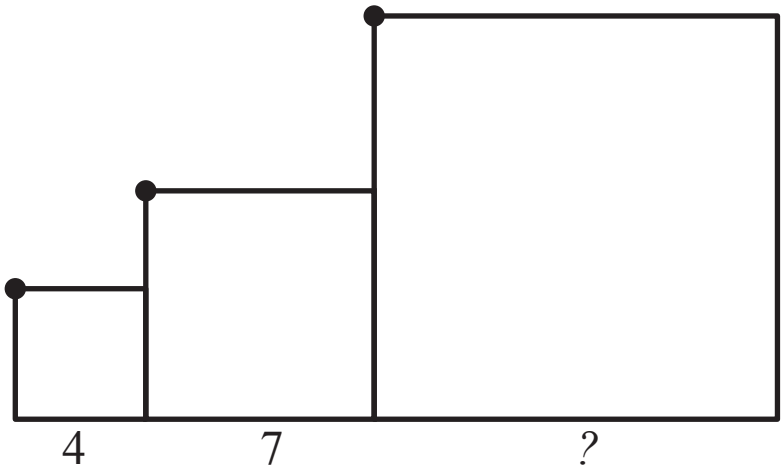
Problème de la semaine

Problème D

Grand, très grand, plus grand

Trois carrés sont placés l'un à côté de l'autre, comme le démontre le diagramme. Le côté du plus petit carré mesure 4 unités, celui du deuxième carré mesure 7 unités, mais la longueur du côté du plus grand carré est inconnue. Cependant, le coin supérieur gauche de chaque carré se trouve sur une même ligne droite.

Détermine la longueur du côté du plus grand carré.



Problem of the Week

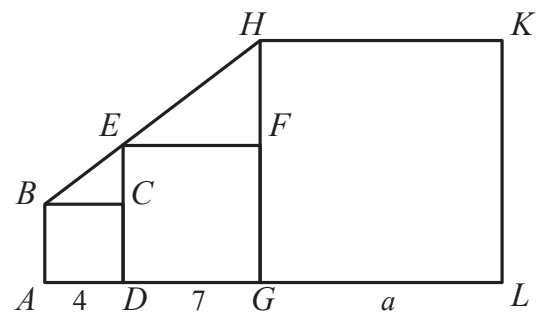
Problem D and Solution

Big, Bigger, Biggest

Problem

Three squares are placed beside each other as shown. The smallest square has side length 4 units, the middle-sized square has side length 7 units, but the side length of the largest square is unknown. However, the top left corner of each of the three squares lies on a straight line. Determine the side length of the largest square.

Solution



Label the vertices as shown on the diagram. Draw line segment BH through E. Let a represent the side length of the larger square.

In Solution 1, we will solve the problem by calculating the slope of BH.

In Solution 2, we will solve the problem using similar triangles.

In Solution 3, we will place the diagram on the xy-plane and solve the problem using analytic geometry.

Solution 1

The slope of a line is equal to its rise divided by its run.

If we look at the line segment from B to E, $BC = 4$ and $CE = DE - DC = 7 - 4 = 3$. Therefore, slope $BE = \frac{CE}{BC} = \frac{3}{4}$.

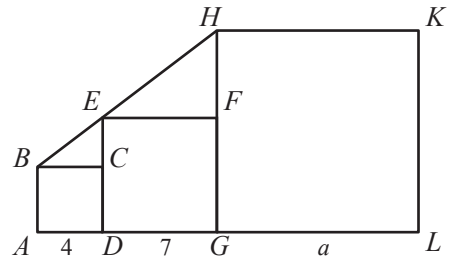
If we look at the line segment from E to H, $EF = 7$ and $FH = GH - GF = a - 7$. Therefore, slope $EH = \frac{FH}{EF} = \frac{a-7}{7}$.

Since B, E and H lie on a straight line, slope BE must equal slope EH. Therefore,

$$\begin{aligned} \text{slope } BE &= \text{slope } EH \\ \frac{3}{4} &= \frac{a-7}{7} \\ 4(a-7) &= 3(7) \\ 4a - 28 &= 21 \\ 4a &= 49 \end{aligned}$$

$\therefore a = \dfrac{49}{4}$ and the side length of the larger square is $\dfrac{49}{4}$ units.

Solution 2



Consider $\triangle BCE$ and $\triangle EFH$. We will first show that $\triangle BCE \sim \triangle EFH$.

Since $ABCD$ is a square, $\angle BCD = 90^\circ$.
Therefore, $\angle BCE = 180^\circ - \angle BCD = 180^\circ - 90^\circ = 90^\circ$.

Since $DEFG$ is a square, $\angle EFG = 90^\circ$.
Therefore, $\angle EFH = 180^\circ - \angle EFG = 180^\circ - 90^\circ = 90^\circ$.
Thus, $\angle BCE = \angle EFH$.

Since $ABCD$ and $DEFG$ are squares and AG is a straight line, BC is parallel to EF.
Therefore, $\angle EBC$ and $\angle HEF$ are corresponding angles and so $\angle EBC = \angle HEF$.

Since the angles in a triangle add to 180°, then we must also have $\angle BEC = \angle EHF$.

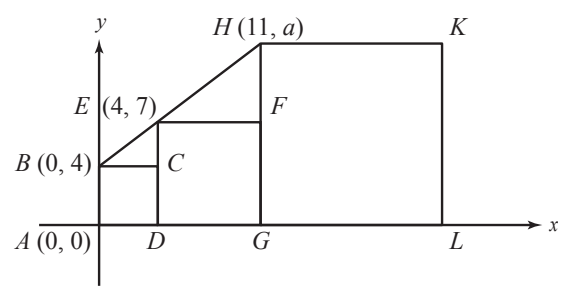
Therefore, $\triangle BCE \sim \triangle EFH$, by Angle−Angle−Angle Triangle Similarity.

Since $\triangle BCE \sim \triangle EFH$, corresponding side lengths are in the same ratio. In particular,

$$\begin{aligned}
\frac{EC}{BC} &= \frac{HF}{EF} \\
\frac{DE - DC}{BC} &= \frac{GH - GF}{EF} \\
\frac{7-4}{4} &= \frac{a-7}{7} \\
\frac{3}{4} &= \frac{a-7}{7} \\
4(a-7) &= 3(7) \\
4a - 28 &= 21 \\
4a &= 49
\end{aligned}$$

$\therefore a = \dfrac{49}{4}$ and the side length of the larger square is $\dfrac{49}{4}$ units.

Solution 3



We proceed by placing the diagram on the xy-plane with A at $(0,0)$ and AL along the x-axis.
The coordinates of B are $(0,4)$, the coordinates of E are $(4,7)$, and the coordinates of H are $(11,a)$.
Let's determine the equation of the line through B, E, H.
Since this line passes through $(0,4)$, it has y-intercept 4.
Since the line passes through $(0,4)$ and $(4,7)$, it has slope $= \frac{7-4}{4-0} = \frac{3}{4}$.
Therefore, the equation of the line is $y = \frac{3}{4}x + 4$.

Since $H(11,a)$ lies on this line, substituting $x = 11$ and $y = a$ into $y = \frac{3}{4}x + 4$, we have

$$a = \frac{3}{4}(11) + 4 = \frac{33}{4} + 4 = \frac{33+16}{4} = \frac{49}{4}$$

$\therefore$ the side length of the larger square is $\dfrac{49}{4}$ units.

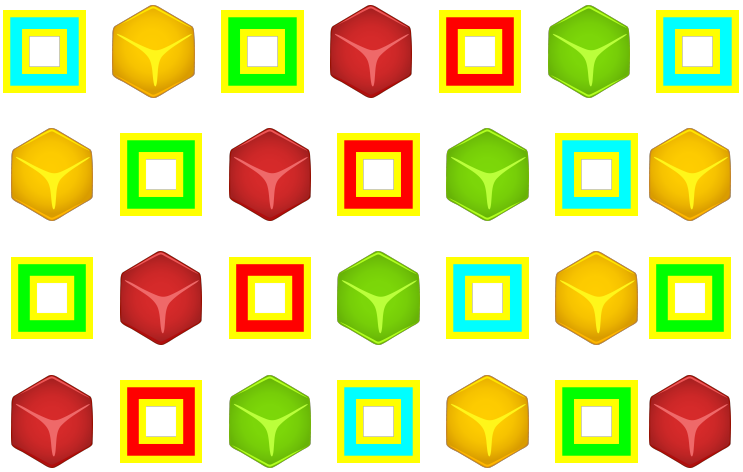
Problème de la semaine

Problème D

Peux-tu répéter ça plus tard?

Si on convertit la fraction $\dfrac{1}{70\ 000\ 000}$ en nombre décimal, quel nombre se trouve-t-il à la 2018e position après la virgule ?

$\vdots$



$\vdots$

Problem of the Week

Problem D and Solution

Can You Repeat That A Little Later?

Problem

When $\dfrac{1}{70\ 000\ 000}$ is written as a decimal, what digit occurs in the 2018th place after the decimal point?

Solution

Notice that $\dfrac{1}{70\ 000\ 000} = \dfrac{1}{10\ 000\ 000} \times \dfrac{1}{7} = 0.000\,000\,1 \times \dfrac{1}{7}$.

Also, note that $\frac{1}{7} = 0.\overline{142857}$. That is, when $\frac{1}{7}$ is written as a decimal, the digits after the decimal point occur in repeating blocks of the 6 digits 142857.

Therefore,

$$\frac{1}{70\ 000\ 000} = 0.000\,000\,1 \times \frac{1}{7} = 0.000\,000\,1 \times 0.\overline{142\,857} = 0.000\,000\,0\overline{1\,428\,57}.$$

That is, when $\frac{1}{70\ 000\ 000}$ is written as a decimal, the digits after the decimal point will be seven 0's followed by repeating blocks of the six digits 142 857.

We see the decimal representation of $\frac{1}{70\ 000\ 000}$ has the same repetition as that for $\frac{1}{7}$, but the pattern is shifted over 7 places. Therefore, the 2018th digit after the decimal point when $\frac{1}{70\ 000\ 000}$ is written as a decimal is the same as the $(2018 - 7) = 2011^{\text{th}}$ digit after the decimal point when $\frac{1}{7}$ is written as a decimal.

Since $\dfrac{2011}{6} = 335\dfrac{1}{6}$, then the 2011th digit after the decimal point occurs after 335 blocks of the repeating digits have been used. In 335 blocks of six digits, there are $335 \times 6 = 2010$ digits in total.

Therefore, the 2011th digit is one digit into the 336th block of repeating digits, so it must be a 1.

The 2018th digit after the decimal point in the decimal representation of $\frac{1}{70\ 000\ 000}$ is the same as the 2011th digit after the decimal point in the decimal representation of $\frac{1}{7}$ and is therefore a 1.

Problème de la semaine

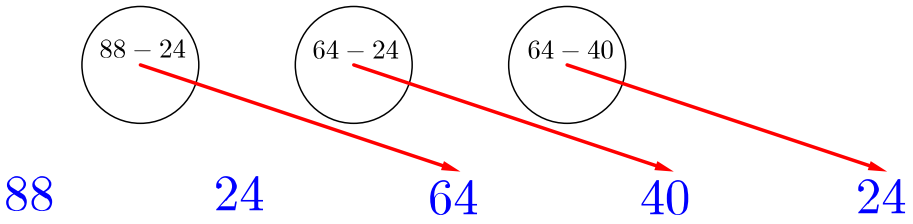
Problème D

Une différence différente

La différence *non-négative* entre deux nombres a et b est la plus grande valeur entre 0, $b - a$ ou $a - b$. Par exemple, la différence non-négative entre 24 et 64 est 40.

Dans la suite $88, 24, 64, 40, 24, \cdots$, chaque terme, après le deuxième terme, est obtenu en évaluant la différence non-négative entre les deux termes précédents. Le diagramme ci-dessous illustre comment obtenir chaque terme de la suite, en commençant par le troisième.

Détermine la somme des 1000 premiers termes de cette suite.



Problem of the Week

Problem D and Solution

A Different Difference

Problem

The *non-negative difference* between two numbers a and b is $b-a$ or $a-b$, whichever is greater than or equal to zero. For example, the non-negative difference between 24 and 64 is 40. In a sequence that begins $74, 60, 14, 46, 32, \cdots$, each number after the second number is obtained by finding the non-negative difference between the previous 2 numbers. Determine the sum of the first 1300 numbers in the sequence.

Solution

We will start by generating more terms of the sequence in an attempt to find a pattern.

Using the rule for creating the sequence, we obtain

$$74, 60, 14, 46, 32, 14, 18, 4, 14, 10, 4, 6, 2, 4, 2, 2, 0, 2, 2, 0, 2, 2, 0, 2, 2, 0, \cdots .$$

The first 14 terms of the sequence have no apparent pattern. The values of the 15th, 16th and 17th terms repeat as the sequence is extended. So the 15th term, 18th term, 21st term, and so on, all equal the first 2 in the string 2,2,0.

The first 14 terms of the sequence are followed by n groups of 2,2,0. What is the value of n? We want a total of 1300 terms. If we remove the first 14 terms, we require $1300 - 14 = 1286$ more terms. If we divide 1286 by 3 we are able to determine how many complete strings of 2,2,0 are needed. The result is $1286 \div 3 = 428\frac{2}{3}$. This tells us that we need 428 complete copies of 2,2,0 and $\frac{2}{3}$ of a copy of 2,2,0, namely 2,2.

The required sum is the sum of the first 14 terms plus $428 \times (2+2+0) + 2 + 2$. The sum of the first 14 terms is 302. The sum of the first 1300 terms of the sequence is $302 + 428 \times 4 + 4 = 2018$.

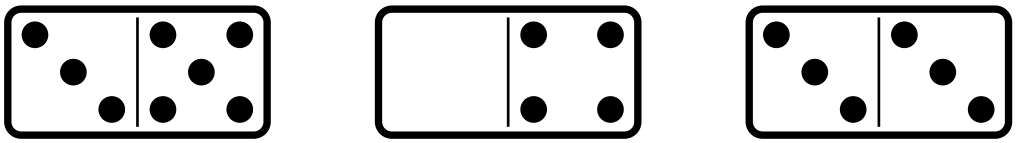
The sum of the first 1300 terms of the sequence is 2018.

Problème de la semaine

Problème D

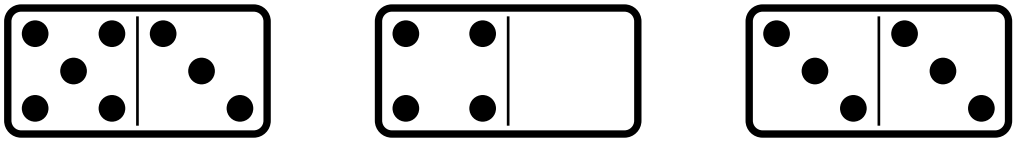
Il manque quelque chose

Un domino est une tuile rectangulaire qu'une ligne verticale divise en deux parties carrées égales. Chaque partie contient un nombre de points ou peut être vide.



Le domino de gauche est un domino $[3, 5]$, puisqu'il a 3 points sur un carré et 5 points sur l'autre. Le domino au milieu est un domino $[0, 4]$, puisqu'une de ses parties carrées est vide et l'autre contient 4 points. Le domino de droite est un domino $[3, 3]$, puisqu'il contient 3 points sur un de ses carrés et 3 points sur l'autre.

Les parties carrées d'un domino peuvent être inversées en tournant le domino de 180°:

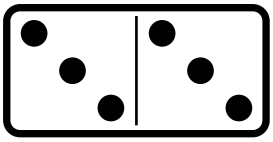


Le domino de gauche est un domino $[5, 3]$. Cependant, puisque les parties carrées de chaque tuile ci-dessus ont été inversées, $[5, 3]$ et $[3, 5]$ représentent le même domino. Le domino du milieu est un domino $[4, 0]$. Noter bien que $[4, 0]$ est équivalent au domino $[0, 4]$.

Une collection-à-n-chiffres de dominos consiste de toutes les tuiles dont les parties carrées contiennent un nombre de points de 0 à n, en excluant toules les tuiles équivalentes.

Par exemple, une collection-à-2-chiffres consiste donc des 6 dominos suivants :
$[0, 0], [0, 1], [0, 2], [1, 1], [1, 2], [2, 2]$.
(À noter que les dominos $[1, 0]$, $[2, 0]$ et $[2, 1]$ sont exclus de cette collection puisqu'ils sont équivalents aux dominos $[0, 1]$, $[0, 2]$ and $[1, 2]$).

Dominic et Dimitri considèrent une collection-à-6-chiffres de dominos: mais ils font tomber une des tuiles de cette collection. Puis, ils séparent les dominos qui restent dans cette collection en deux piles qui ne sont pas égales. Dimitri compte tous les points sur les dominos dans la première pile. Il compte un total de 91 points. Dominic compte tous les points des dominos dans la seconde pile. Il en compte 67 en tout. Dimitri et Dominic ont fait attention que leur pile ne contenait aucuns dominos équivalents. Quel est le domino tombé?



Problem of the Week
Problem D and Solution
Something's Missing

Problem

Dominic and Dmitri divide a 6-set of dominoes into two unequal piles. They then realize that one domino is missing from the set. Dimitri counts all of the pips on the dominoes in the first pile. He counts that there are a total of 91 pips. Dominic counts all of the pips on the dominoes in the second pile. He counts that there are a total of 67 pips. Dominic also notes that all the double dominoes are accounted for. Which domino is missing from the set?

Solution

We first determine which dominoes are in a 6-set and calculate the total number of pips on all of the dominoes in the set. In a 6-set of dominoes, the number of pips on each end of a domino tile can range from 0 to 6. Since rotating a domino tile does not change the domino, let's orient each tile so that the smaller number on non double dominoes is always on the left end of the tile.

For each possible number on the left side of the domino, let's examine the possible numbers that can occur on the right side and compile this information in a table.

Number on Left	Possible Numbers on Right	Sum of Pips on Dominoes
0	0, 1, 2, 3, 4, 5, 6	$1 + 2 + 3 + 4 + 5 + 6 = 21$
1	1, 2, 3, 4, 5, 6	$6(1) + 1 + 2 + 3 + 4 + 5 + 6 = 27$
2	2, 3, 4, 5, 6	$5(2) + 2 + 3 + 4 + 5 + 6 = 30$
3	3, 4, 5, 6	$4(3) + 3 + 4 + 5 + 6 = 30$
4	4, 5, 6	$3(4) + 4 + 5 + 6 = 27$
5	5, 6	$2(5) + 5 + 6 = 21$
6	6	$1(6) + 6 = 12$

Therefore, the total number of pips on the dominoes in a 6-set is

$$21 + 27 + 30 + 30 + 27 + 21 + 12 = 168.$$

Since the total number of pips in the first pile is 91, and the total number of pips in the second pile is 67, the sum is 158. That leaves a total of 10 pips on the missing tile.

In a 6-set of dominoes, the only tiles with 10 pips are $[5, 5]$ and $[4, 6]$. Since all the double dominoes are present, then the $[5, 5]$ is present. Therefore, the missing tile must be the $[4, 6]$ tile.

Problème de la semaine

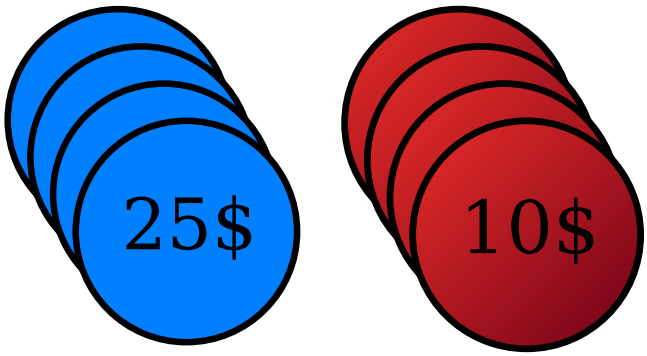
Problème D

Des piles et des piles

On te donne le même nombre de jetons rouges que de jetons bleus pour résoudre le problème suivant. Un jeton rouge a une valeur de 10$ et un jeton bleu a une valeur de 25$.

En utilisant uniquement ces jetons rouges et ces jetons bleus, crée des piles de jetons de façon à ce que chaque pile ait une valeur de 750$. Deux piles quelconques ne peuvent avoir le même nombre de jetons rouges et chaque pile doit avoir au moins un jeton rouge et un jeton bleu.

Détermine le nombre maximum de piles différentes qui peuvent être ainsi créées.



Problem of the Week
Problem D and Solution
Stack 'em High

Problem

You have as many red coins and blue coins as you need to complete this problem. Each red coin is worth \$10 and each blue coin is worth \$25. Using only red coins and blue coins, create stacks of coins so that each stack has a total value of \$750. No two stacks can have the same number of red coins and each stack must contain at least one red coin and one blue coin. Determine the maximum number of different stacks of coins that can be made.

Solution

Solution 1

If no red coins were required, you would need $\$750 \div \$25 = 30$ blue coins. Therefore, at most $30 - 1 = 29$ blue coins would be required for any one stack.

If you were to use an odd number of blue coins, then the total value of the blue coins would be some number with ones digit 5. The value of the required red coins would then be \$750 minus the value of the blue coins, producing a difference whose ones digit is also 5. But each red coin is worth \$10 and no combination of red coins could produce a total whose ones digit is 5. Therefore, it is not possible to have an odd number of blue coins.

So the possible number of blue coins in any stack is an even integer between 1 and 29. There are 14 even numbers in this range. Since any even multiple of \$25 produces a number whose units digit is 0 and \$750 minus the value of the blue coins would then also has a units digit 0, there exists some number of red coins that will produce each possible difference.

Therefore, the maximum number of different stacks of coins is 14. (In the second solution the possibilities will be listed.)

Solution 2

In this solution, a more algebraic argument will be presented.

Let b represent the number of blue coins and r represent the number of red coins. Since each stack must contain at least 1 coin of each colour and we must use whole coins, both b and r are integers such that $b \geq 1$ and $r \geq 1$.

Since each blue coin is worth \$25, the total value of the blue coins is $25b$. Since each red coin is worth \$10, the total value of the red coins is $10r$. The total value of each stack is \$750 so $25b + 10r = 750$.

Dividing both sides of the equation by 5, we obtain $5b + 2r = 150$. (1)

Rearranging equation (1) to isolate $5b$, we obtain $5b = 150 - 2r$. (2)

Each term on the right side of equation (2) is even so the difference $150 - 2r$ will also be even. The value of the left side of equation (2) must then also be even. This can only be accomplished for even values of b. (An odd integer multiplied by 5 produces an odd integer but an even integer multiplied by 5 produces an even integer.)

Since each stack must contain at least one coin of each colour, letting $r = 1$ in equation (2) will generate the largest possible value of b. When $r = 1$, equation (2) becomes $5b = 150 - 2 = 148$ and $b = 29.6$. Since b is an integer, b is at most 29.

We now know that b is an even integer such that $1 \leq b \leq 29$. There are 14 even integers in this set of numbers, namely $2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26$, and 28.

Although not required, we can calculate the corresponding r values for each of the b values.

Written as a set of ordered pairs (b, r), we have

$$(2, 70),\ (4, 65),\ (6, 60),\ (8, 55),\ (10, 50),\ (12, 45),\ (14, 40)$$
$$(16, 35), (18, 30),\ (20, 25),\ (22, 20),\ (24, 15),\ (26, 10), \text{and } (28, 5).$$

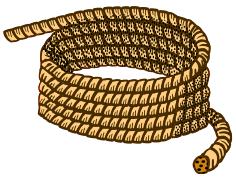
The restriction stating that no two stacks could have the same number of red coins has clearly been satisfied.

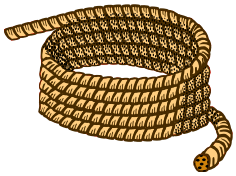
Problème de la semaine

Problème D

Encordé

Une corde de 200 cm de longueur est coupée en quatre morceaux. Trois morceaux sont utilisés pour créer des triangles équilatéraux identiques dont la longueur des côtés est un nombre entier. Le quatrième morceau est utilisé pour former un carré dont la longueur des côtés est aussi un nombre entier. Détermine tous les triangles et le carré possibles en donnant explicitement toutes les longueurs de leurs côtés.





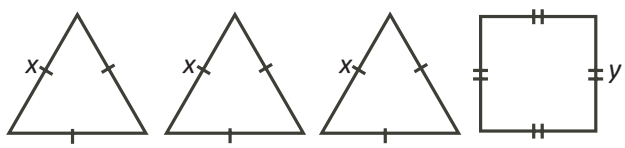
Problem of the Week
Problem D and Solution
Roped In

Problem

A rope of length 200 cm is cut into four pieces. Three of the pieces are used to form identical equilateral triangles with integer side lengths. The fourth piece is used to form a square with integer side lengths. Determine all possible side lengths for each triangle and square.

Solution

Let x represent the integer side length of each equilateral triangle and let y represent the integer side length of the square.



The perimeter of each figure is the length of the piece of rope used to form it. For each triangle, the length of rope is $3x$ and for the square the length of rope is $4y$. The total rope used is $3(3x) + 4y = 9x + 4y$. But the length of the rope is 200 cm. Therefore,

$$\begin{aligned} 9x + 4y &= 200 \\ 9x &= 200 - 4y \\ x &= \frac{4(50-y)}{9} \end{aligned}$$

Since both x and y are integers, $4(50-y)$ must be a multiple of 9. But 4 is not divisible by 9, so $50 - y$ must be divisible by 9. There are five multiples of 9 between 0 and 50, namely 9, 18, 27, 36, and 45. So $50 - y = \{9, 18, 27, 36, 45\}$ and it follows that $y = \{41, 32, 23, 14, 5\}$. The corresponding values of x are computed in the chart below.

y	$4y$	$200-4y$	$x = \dfrac{200-4y}{9}$
41	164	36	4
32	128	72	8
23	92	108	12
14	56	144	16
5	20	180	20

When the side length of the square is 41 cm, the side length of each triangle is 4 cm; when the side length of the square is 32 cm, the side length of each triangle is 8 cm; when the side length of the square is 23 cm, the side length of each triangle is 12 cm; when the side length of the square is 14 cm, the side length of each triangle is 16 cm; and when the side length of the square is 5 cm, the side length of each triangle is 20 cm.

Problème de la semaine

Problème D

Une longue escale

La vitesse moyenne d'un vol d'un certain avion, y compris le décollage et l'atterrissage, est de 500 km/h.

L'avion vole entre Toronto et Thunder Bay en 1 heure et 40 minutes, fait une courte escale, puis retourne à Toronto. Le retour suit la même trajectoire et requiert autant de temps que l'allée.

Si l'escale à Thunder Bay diminue la vitesse moyenne de l'avion à 425 km/h pour le voyage aller-retour au complet, combien de temps a duré l'escale?

Problem of the Week
Problem D and Solution
Lengthy Layover

Problem

Including takeoff and landing, a plane has an average speed of 500 km/h while in flight. The plane travels from Toronto to Thunder Bay in 1 hour 40 minutes, has a brief layover, then returns to Toronto. The return flight is by the same route and in the same amount of time. If the wait in Thunder Bay reduces the average speed to 425 km/h for the entire two-way trip, how long was the layover?

Solution

Let t be the length of the layover, in hours.

The plane travels from Toronto to Thunder Bay in 1 hour 40 minutes at a speed of 500 km/h. Using the formula distance = speed × time, the distance from Toronto to Thunder Bay must be $500\frac{\text{km}}{\text{h}} \times 1\frac{2}{3}\text{ h} = 500 \times \frac{5}{3} = \frac{2500}{3}$ km.

Therefore, for the two-way trip, the plane travels $2 \times \frac{2500}{3} = \frac{5000}{3}$ km.

The length of time of the entire two-way trip is the time of the two flights plus the layover time. Therefore, the total length of time of the trip is $\frac{5}{3} + \frac{5}{3} + t = \frac{10}{3} + t$ hours.

Since the average speed of the entire two-way trip is 425 km/h, using the formula distance = speed × time, we have

$$\begin{aligned}
\frac{5000}{3} &= 425 \times \left(\frac{10}{3} + t\right) \\
\frac{10}{3} + t &= \frac{5000}{3 \times 425} \\
t &= \frac{200}{51} - \frac{10}{3} \\
&= \frac{200}{51} - \frac{170}{51} \\
&= \frac{10}{17}
\end{aligned}$$

Therefore, the layover was $\dfrac{10}{17}$ hours, or approximately 35 minutes.

Problème de la semaine

Problème D

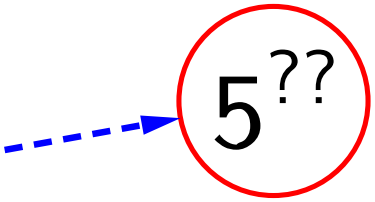
La puissance de 5

Soit N le produit des 1000 premiers nombres entiers positifs noté par 1000! (qui se lit "*factorielle de N*"): on définit donc N par

$$N = 1000! = 1000 \times 999 \times 998 \times 997 \times \cdots \times 3 \times 2 \times 1.$$

N est divisible par $5, 25, 125, 625, \cdots$. Chacun de ces facteurs est une puissance de 5, c'est-à-dire, $5 = 5^1,\ 25 = 5^2,\ 125 = 5^3,\ 625 = 5^4,$ etc.

Détermine la plus grande puissance de 5 qui est un facteur de N.



Problem of the Week

Problem D and Solution

The Power of Five

Problem

The number N is the product of the first 1000 positive integers and can be written as 1000!. We say, "1000 *factorial*." That is, $N = 1000! = 1000 \times 999 \times 998 \times 997 \times \cdots \times 3 \times 2 \times 1$. N is divisible by $5, 25, 125, 625, \cdots$. Each of these factors is a power of 5. That is, $5 = 5^1,\ 25 = 5^2,\ 125 = 5^3,\ 625 = 5^4,$ and so on. Determine the largest power of 5 that divides N.

Solution

Solution 1

In order to determine the largest power of 5 that divides N, we need to count the number of times the factor 5 appears in the factorization of N.

First, let's look at the numbers that are divisible by 5 in $N!$. Each of the numbers $\{5, 10, 15, 20, \ldots, 990, 995, 1000\}$ contains a factor of 5. That is a total of $\frac{1000}{5} = 200$ factors of 5.

Those numbers that are multiples of 25 will add an additional factor of 5, since $25 = 5 \times 5$. There are $\frac{1000}{25} = 40$ numbers less than or equal to 1000 which are multiples of 25. So we gain another 40 factors of 5 bringing the total to $200 + 40 = 240$.

Those numbers that are multiples of 125 will add an additional factor of 5. This is because $125 = 5 \times 5 \times 5$ and two of the factors have already been counted when we looked at 5 and 25. There are $\frac{1000}{125} = 8$ numbers less than or equal to 1000 which are multiples of 125. So we gain another 8 factors of 5 bringing the total to $240 + 8 = 248$.

Those numbers that are multiples of 625 will add an additional factor of 5. This is because $625 = 5 \times 5 \times 5 \times 5$ and three of the factors have already been counted when we looked at 5, 25 and 125. There is 1 number less than 1000 which is a multiple of 625 (namely, 625). So we gain another factor of 5 bringing the total to $248 + 1 = 249$. So, when N is factored there are 249 factors of 5. Therefore, the largest power of 5 that divides N is 5^{249}.

Solution 2

There are many similarities between solution 1 and the following solution. In this solution we will divide out multiples of 5 until there are none left.

1. In 1000!, there are $\frac{1000}{5} = 200$ multiples of 5, namely, $\{5, 10, 15, \cdots, 990, 995, 1000\}$. No other numbers from 1 to 1000 are divisible by 5. If we divide each number in this first list by 5, we obtain the second list $\{1, 2, 3, \cdots, 198, 199, 200\}$.
2. This second list contains $\frac{200}{5} = 40$ more multiples of 5, namely, $\{5, 10, 15, \cdots, 190, 195, 200\}$. No other numbers from 1 to 200 are divisible by 5. If we divide each number in this second list by 5, we obtain the third list $\{1, 2, 3, \cdots, 38, 39, 40\}$.
3. This third list contains $\frac{40}{5} = 8$ more multiples of 5, namely, $\{5, 10, 15, 20, 25, 30, 35, 40\}$. No other numbers from 1 to 40 are divisible by 5. If we divide each number in the third list by 5, we obtain the fourth list $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
4. This fourth list contains 1 more multiples of 5, namely the number 5. No other numbers from 1 to 8 are divisible by 5.

In total, there are $200 + 40 + 8 + 1 = 249$ factors of 5 in 1000!. Therefore, the largest power of 5 that divides N is 5^{249}.

An interpretation of what has happened is in order. When we created the first list of multiples of 5, we discovered that there were 200 numbers from 1 to 1000 that were divisible by 5. When we created the second list of multiples of 5, we were actually counting the 40 numbers from 1 to 1000 that were divisible by 25. When we created the third list of multiples of 5, we were actually counting the 8 numbers from 1 to 1000 that were divisible by 125. And finally, when we created the fourth list of multiples of 5, we were actually counting the 1 number from 1 to 1000 that was divisible by 625.

NOTE: We define int(n) as the largest integer less than or equal to n. For example, int(42)=42 and int(37.6)=37. We can count the number of numbers divisible by various powers of 5 as follows:

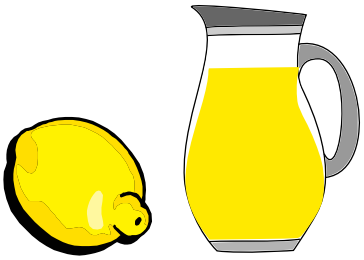
$$\begin{aligned}
\text{\# of Factors of 5 in 1000!} &= \text{int}\left(\tfrac{1000}{5}\right) + \text{int}\left(\tfrac{1000}{25}\right) + \text{int}\left(\tfrac{1000}{125}\right) + \text{int}\left(\tfrac{1000}{625}\right) \\
&= \text{int}(200) + \text{int}(40) + \text{int}(8) + \text{int}(1.6) \\
&= 200 + 40 + 8 + 1 \\
&= 249
\end{aligned}$$

Problème de la semaine

Problème D

Concentré

Un contenant de jus contient un mélange de concentré de jus de citron et d'eau. Lorsqu'un litre d'eau est ajouté au contenant, le rapport entre la masse du concentré de jus et celle de l'eau est de 1 : 2. Lorsqu'un litre de concentré de jus est ajouté à ce nouveau mélange, le rapport devient de 2 : 3. Détermine le rapport entre la masse du concentré de jus et celle de l'eau, dans le mélange original.



Problem of the Week
Problem D and Solution
Concentrate

Problem

A large jug of lemonade contains a mixture of lemon juice concentrate and water. When 1 litre of water is added to the jug, the ratio, by volume, of juice concentrate to water is $1:2$. When 1 litre of juice concentrate is added to the new mixture, the ratio becomes $2:3$. Find the original ratio, by volume, of lemon juice concentrate to water in the jug.

Solution

Let j be the amount of juice, in litres, in the original mixture.
Let w be the amount of water, in litres, in the original mixture.

When 1 litre of water is added, the ratio of juice concentrate to water is $1:2$. This tells us that

$$\frac{j}{w+1} = \frac{1}{2} \qquad (1)$$

Simplifying (1), we obtain $w+1=2j$ and $w=2j-1$ follows. (2)

When 1 litre of juice concentrate is added to the new mixture, the ratio becomes $2:3$. This tells us that

$$\frac{j+1}{w+1} = \frac{2}{3} \qquad (3)$$

From (2), since $w=2j-1$, we can substitute for w in (3), obtaining:

$$\begin{aligned}
\frac{j+1}{2j-1+1} &= \frac{2}{3} \\
\frac{j+1}{2j} &= \frac{2}{3} \\
2(2j) &= 3(j+1) \\
4j &= 3j+3 \\
j &= 3
\end{aligned}$$

Substituting $j=3$ in $w+1=2j$ we obtain $w+1=6$ and $w=5$ follows.

So there are 3 litres of juice concentrate in the container and 5 litres of water. The ratio of juice concentrate to water is $3:5$.

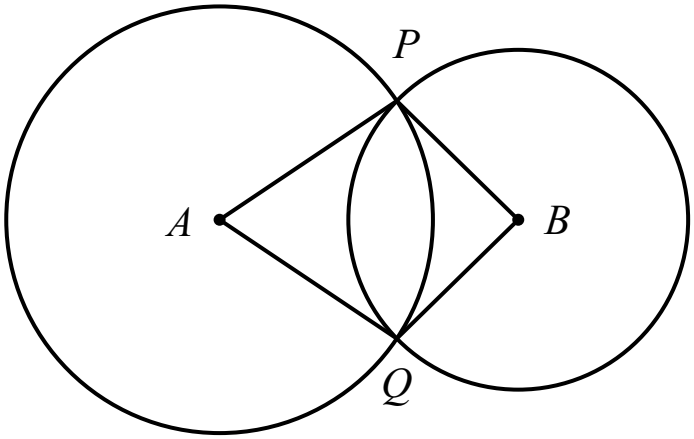
Problème de la semaine

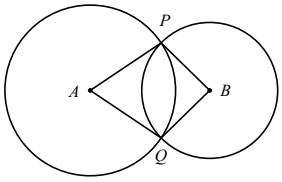
Problème D

Quelle est la relation?

Deux cercles dont les centres sont, respectivement, les points A et B, se croisent aux points P et Q de sorte que $\angle PAQ = 60^\circ$ et $\angle PBQ = 90^\circ$.

Quelle est la relation entre l'aire du cercle ayant pour centre le point A et l'aire de celui dont le centre est le point B?





Problem of the Week

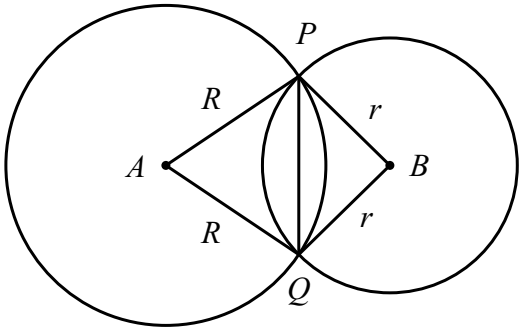
Problem D and Solution

How are We Related?

Problem

Two circles, one with centre A and one with centre B, intersect at points P and Q such that $\angle PAQ = 60^\circ$ and $\angle PBQ = 90^\circ$. How is the area of the circle with centre A related to the area of the circle with centre B?

Solution



Let R be the radius of the circle with centre A and r be the radius of the circle with centre B.

Join P to Q.

We will determine the length of PQ in terms of R and then in terms of r in order to find a relationship between R and r.

Consider $\triangle APQ$. Since $AP = AQ = R$, $\triangle APQ$ is isosceles and so $\angle APQ = \angle AQP$. Since $\angle PAQ = 60^\circ$, $\angle APQ = \angle AQP = \frac{180^\circ - 60^\circ}{2} = 60^\circ$. Therefore, $\triangle APQ$ is equilateral and $PQ = AP = AQ = R$.

Consider $\triangle BPQ$. We are given that $\angle PBQ = 90^\circ$. Therefore, $\triangle BPQ$ is a right-angled triangle. The Pythagorean theorem tells us that $PQ^2 = BP^2 + BQ^2 = r^2 + r^2 = 2r^2$.

We have $PQ = R$ and $PQ^2 = 2r^2$. Therefore, $R^2 = 2r^2$.

The area of the circle with centre B and radius r is πr^2.

The area of the circle with centre A and radius R is

$$\pi R^2 = \pi(2r^2) = 2(\pi r^2) = 2 \times (\text{ the area of the circle with centre } B).$$

Therefore, the area of the circle with centre A is twice the area of the circle with centre B.

Problème de la semaine

Problème D

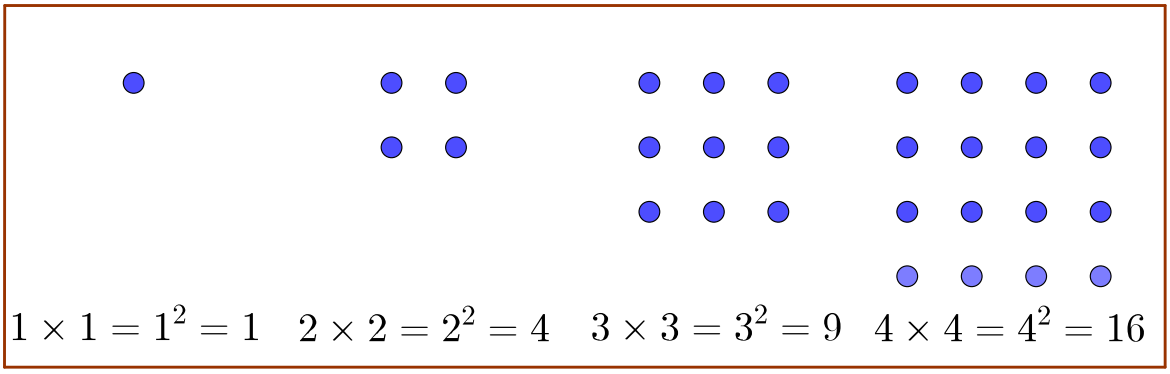
Ce nombre, c'est la perfection

Un *carré parfait* est un nombre entier qui peut être exprimé comme le produit de deux nombres entiers égaux. 25 est un carré parfait puisqu'il peut être exprimé comme le produit 5×5 ou 5^2.

Multiplions par le même nombre entier positif n, tous les nombres entiers positifs pairs compris entre 2 et 1600, inclusivement. Ces produits sont ensuite additionnés ensemble. Le résultat de cette somme est alors un carré parfait.

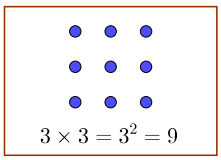
Détermine la valeur du plus petit nombre entier positif n pour lequel ce résultat est vérifié.

Les quatre premiers carrés parfaits



Le résultat suivant peut être utile, mais n'est pas indispensable, pour résoudre le problème: la somme S des nombres entiers positifs compris entre 1 et n inclusivement, peut être calculé en utilisant la formule $S = \frac{n \times (n+1)}{2}$.
Par exemple,

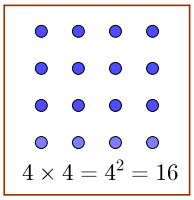
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{10 \times 11}{2} = 55.$$



Problem of the Week

Problem D and Solution

This Number Makes It Perfect



Problem

The positive even integers 2 to 1600, inclusive, are each multiplied by the same positive integer, n. All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer n that makes this true.

Solution

What does the prime factorization of a perfect square look like? Let's look at a few examples: $9 = 3^2$, $36 = 6^2 = 2^2 3^2$, and $129\,600 = 360^2 = 2^6 5^2 3^4$. Notice that the exponent on each of the prime factors in the prime factorization in each of the three examples is an even number.

The positive integer n is the smallest positive integer such that

$$2n + 4n + 6n + \cdots + 1596n + 1598n + 1600n \qquad (1)$$

is a perfect square.

Factoring (1), we obtain

$$\begin{aligned} & 2n(1 + 2 + 3 + \cdots + 798 + 799 + 800) \\ = \; & 2n\left(\frac{800 \times 801}{2}\right) \\ = \; & n(800)(801) \qquad (2) \\ = \; & n[(2)(2)(2)(2)(2)(5)(5)][(3)(3)(89)] \\ = \; & n(2^5)(5^2)(3^2)(89) \qquad (3) \end{aligned}$$

In going from (2) to (3), we have expressed the 800×801 as the product of prime factors.

We need to determine what additional factors are required to make the quantity in (3) a perfect square such that n is as small as possible. In order for the exponent on each prime in the prime factorization to be even, we need n to be $89 \times 2 = 178$. Then the quantity in (3) becomes

$$n(2^5)(5^2)(3^2)(89) = (2)(89)(2^5)(5^2)(3^2)(89) = (2^6)(5^2)(3^2)(89^2) = \left[(2^3)(5)(3)(89)\right]^2,$$

a perfect square.

Therefore, the smallest positive integer is 178 and the perfect square is

$$178 \times 800 \times 801 = 114\,062\,400 = (10\,680)^2.$$

Problème de la semaine

Problème D

Des sauts et des bonds

La majorité des personnes croit qu'une année consiste en 365 jours. En réalité, une année contient un peu plus que 365 jours. Pour tenir compte de ce temps supplémentaire, nous utilisons des années bissextiles qui comportent un jour de plus. La règle suivante est utilisée pour déterminer si une année est bissextile :

Une année est une année bissextile si:

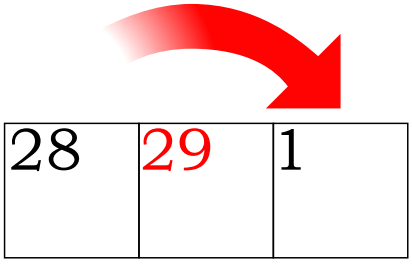
- elle est un multiple de 4, et
- elle n'est pas un multiple de 100, à moins d'être aussi un multiple de 400.

Par exemple, 2018 n'est pas une année bissextile puisque 2018 n'est pas un multiple de 4. Cependant, 2016 était une année bissextile puisque 2016 est un multiple de 4, mais pas un multiple de 100.

La deuxième partie de cette règle concerne seulement la première année de chaque nouveau siècle.

Par exemple, les années 2000 et 2400 sont des années bissextiles puisque les deux sont des multiples de 400. Cependant, les années 2100, 2200 et 2300 ne sont pas des multiples de 400, donc elles ne sont pas des années bissextiles.

Une année supérieure à 2000 est choisie au hasard. Quelle est la probabilité qu'elle soit une année bissextile?



Problem of the Week

Problem D and Solution

Leaps and Bounds

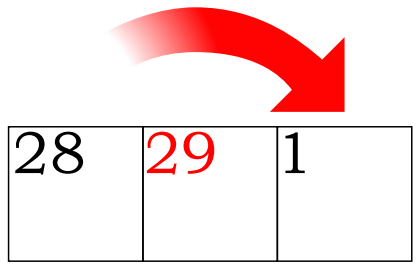
Problem

Most people believe a year is equivalent to 365 days. In actuality, it is slightly more than 365 days. To account for this extra time, we use leap years, which are years containing one extra day. The following rule is used to determine if a year is a leap year:

A year is a leap year if it is

- divisible by four, and
- not divisible by 100, unless it is also divisible by 400.

A year greater than 2000 is chosen at random, what is the probability that it is a leap year?



Solution

The probability of an event occurring is calculated as the number of favourable outcomes divided by the total number of possible outcomes. This is an issue in our problem because the number of years greater than 2000 is infinite. However, in this case, the cycle of leap years repeats every 400 years. For example, since 2044 is a leap year so is 2444.

The number of leap years in a cycle can be counted this way.

The multiples of 4 in a 400 year cycle is $\frac{400}{4} = 100$. However, we have counted the multiples of 100 so we need to subtract these multiples which is $\frac{400}{100} = 4$ to get $100 - 4 = 96$. We now need to add back the the multiples of 400 which is $\frac{400}{400} = 1$ to get $96 + 1 = 97$.

Therefore for every 400 year cycle, 97 of these years will be a leap year and the probability of getting a leap year is $\frac{97}{400} = 0.2425$.

Problème de la semaine

Problème D

Sommer

Une suite de nombres entiers positifs est obtenue de la manière suivante : les deux premiers termes sont définis, et chaque terme après le deuxième est obtenu en calculant la somme de tous les termes précédents de la suite.

Par exemple, si les deux premiers termes de la suite sont 2 et 8, les quatre prochains termes seraient $10, 20, 40$ et 80.

Une suite est obtenue de la façon décrite au-dessus, avec comme premier terme le chiffre 3 et comme autre terme dans la suite, le chiffre 3072. Combien de suites de cette forme existe-t-il?

SUMMIT

Problem of the Week

Problem D and Solution

Sum It

Problem

A sequence of non-negative integers is formed in the following way: the first two terms of the sequence are defined, and then each term after the second term is the sum of all previous terms in the sequence. For example, if the first two terms of the sequence were 2 and 8, the next four terms of the sequence would be $10, 20, 40$ and 80.

A sequence is formed as described above such that the first term is 3 and some other term in the sequence is 3072. How many such sequences are there?

Solution

We know how to construct the sequence and we know that it starts with first term 3, but where is the term whose value is 3072?

Can the second term be 3072?

If the first two terms are 3 and 3072, then the third term would be
$3 + 3072 = 3075$.
The fourth term would be $\mathbf{3} + \mathbf{3072} + 3075 = \mathbf{3075} + 3075 = 2(3075) = 6150$.
The fifth term would be
$\mathbf{3} + \mathbf{3072} + \mathbf{3075} + 6150 = \mathbf{6150} + 6150 = 2(6150) = 12\,300$.

We see that we can determine any term beyond the third term by summing all of the previous terms or we can simply double the term immediately before the required term, since that term is the sum of all the preceding terms. (This also means that if any term after the third term is known, then the preceding term is half the value of that term.)

Therefore, there is a sequence with second term 3072. The first 6 terms of this sequence are $3, 3072, 3075, 6150, 12\,300, 24\,600$.

Can the third term be 3072?

Yes, since the third term is the sum of the first two terms and the first term is 3, then the second term is $3072 - 3 = 3069$ and the first 6 terms of the sequence are $3,\ 3069,\ 3072,\ 6144,\ 12\,288,\ 24\,576$.

Continued on the next page.

Can the fourth term be 3072?

Yes, since the fourth term is even, then we can determine the third term to be half of the fourth term, $3072 \div 2 = 1536$, and the second term is $1536 - 3 = 1533$. This sequence starts $3,\ 1533,\ 1536,\ 3072,\ 6144,\ 12\,288$.

Can the fifth term be 3072?

To get from the fifth term to the third term we would divide by 2 twice, or we could divide by 4. If the resulting third term is a non-negative integer greater than or equal to 3, the sequence exists. The third term is $3072 \div 4 = 768$ and the second term would be $768 - 3 = 765$. Yes the sequence exists and it starts $3,\ 765,\ 768,\ 1536,\ 3072,\ 6144$.

We could continue in this way until we discover all possible sequences that are formed according to the given rules with first term 3 and 3072 somewhere in the sequence. However, if we look at the prime factorization of 3072 we see that the highest power of 2 that divides 3072 is 1024 since $3072 = 2^{10} \times 3$. In fact, dividing 3072 by 1024 would produce a third term that would be 3. The second term would then be 0, a non-negative integer, and the resulting sequence would be $3, 0, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536, 3072, 6144, \cdots$.

If we divide 3072 by any integral power of 2 from $2^0 = 1$ to $2^{10} = 1024$, the resulting third term would be an integer greater than or equal to 3, and 3072 would appear in each of these sequences. There are 11 such sequences. The number 3072 would appear somewhere from term 3 to term 13 in the acceptable sequence. However, 3072 can also appear as the second term so there are a total of 12 possible sequences.

Could 3072 be the fourteenth term? From the fourteenth term to the third term we would need to divide 3072 by 2^{11}. The resulting third term would be $\frac{3}{2}$. This is not a non-negative integer and so the sequence is not possible.

Therefore, there are a total of 12 such sequences.

Problème de la semaine

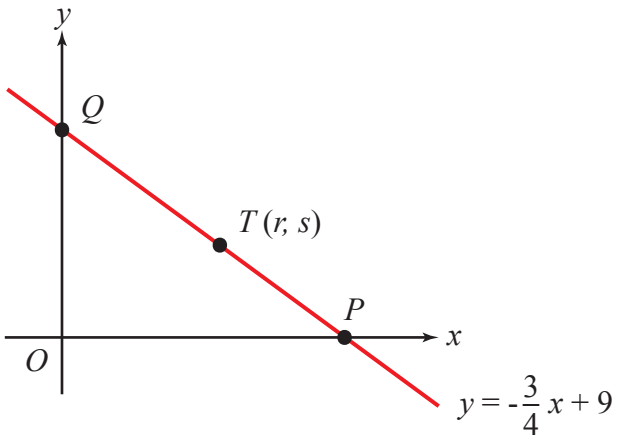
Problème D

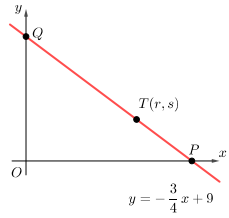
Le point de division

La droite $y = -\frac{3}{4}x + 9$ croise l'axe des x au point P et l'axe des y au point Q.

Le point $T(r, s)$ se situe sur le segment PQ de façon à ce que l'aire de POQ soit trois fois celle de TOP.

Détermine les valeurs de r et s, les coordonnées du point T.





Problem of the Week

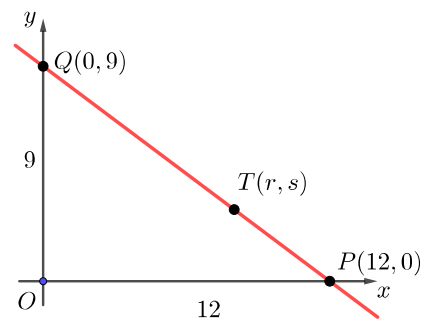
Problem D and Solution

A Point of Division

Problem

The line $y = -\frac{3}{4}x + 9$ crosses the x-axis at P and the y-axis at Q. Point $T(r, s)$ lies on the line segment PQ such that the area of $\triangle POQ$ is three times the area of $\triangle TOP$. Determine the values of r and s, the coordinates of T.

Solution

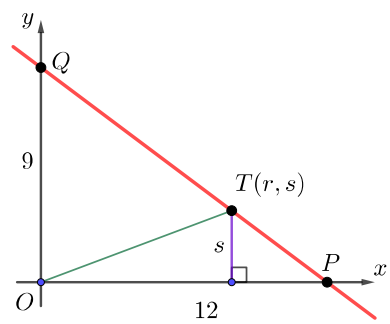


We begin by calculating the coordinates of P and Q, the x- and y-intercepts of the line $y = -\frac{3}{4}x + 9$.

Since the equation of the line is written in the form $y = mx + b$ where b is the y-intercept of the line, the y-intercept is 9 and so the coordinates of Q are $(0, 9)$. To determine the x-intercept, set $y = 0$ to obtain $0 = -\frac{3}{4}x + 9 \implies \frac{3}{4}x = 9 \implies x = 12$. Thus, P has coordinates $(12, 0)$.

We now present two different solutions to the problem.

Solution 1



Since $\triangle POQ$ is a right triangle with base $PO = 12$ and height $OQ = 9$, using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have $\text{area}(\triangle POQ) = \frac{12 \times 9}{2} = 54$.

Since the area of $\triangle POQ$ is three times the area of $\triangle TOP$, $\text{area}(\triangle TOP) = \frac{1}{3}(\text{area}(\triangle POQ)) = \frac{1}{3}(54) = 18$.

$\triangle TOP$ has area 18, base $PO = 12$ and height s. Using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have

$$\begin{aligned} \text{area}(\triangle TOP) &= \frac{PO \times s}{2} \\ 18 &= \frac{12 \times s}{2} \\ 18 &= 6s \\ \therefore\ s &= 3 \end{aligned}$$

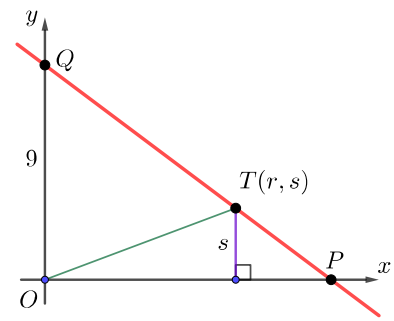
$T(r, s)$ lies on the line $y = -\frac{3}{4}x + 9$ and $s = 3$ so we can substitue $x = r$ and $y = 3$

$$\begin{aligned} 3 &= -\frac{3}{4}r + 9 \\ \frac{3}{4}r &= 6 \\ \therefore\ r &= 8 \end{aligned}$$

Therefore, T is the point $(r, s) = (8, 3)$.

Solution 2

If two triangles have equal bases, the areas of the triangles are proportional to the heights of the triangles.



$\triangle POQ$ and $\triangle TOP$ have the same base, OP.

Since the area of $\triangle POQ$ is 3 times the area of $\triangle TOP$, then the height of $\triangle POQ$ is 3 times the height of $\triangle TOP$. In other words, the height of $\triangle TOP$ is $\frac{1}{3}$ the height of $\triangle POQ$. $\triangle POQ$ has height $OQ = 9$ and $\triangle TOP$ has height s. Therefore, $s = \frac{1}{3}(OQ) = \frac{1}{3}(9) = 3$.

Since $T(r, s)$ lies on the line $y = -\frac{3}{4}x + 9$, we have

$$\begin{aligned} s &= -\frac{3}{4}r + 9 \\ 3 &= -\frac{3}{4}r + 9 \\ \frac{3}{4}r &= 6 \\ \therefore\ r &= 8 \end{aligned}$$

Therefore, T is the point $(r, s) = (8, 3)$.

Note that it was actually unnecessary to find the x-intercept for the second solution as it was never used in the second solution.

For Further Thought:

Find the coordinates of S, another point on line segment QP, so that

$$\text{the area of } \triangle SOQ = \text{ the area of } \triangle TOP,$$

thus creating three triangles of equal area. How are the points Q, S, T, and P related?

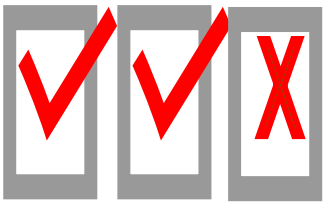
Problème de la semaine

Problème D

Pourcentage d'intérêt

L'information suivante est connue au sujet des résultats d'un test de mathématiques :

- le test possédait trois questions;
- chaque question avait une valeur de 1 point;
- chaque réponse était évaluée, soit correcte ou incorrecte;
- 50% des élèves ont eu les 3 bonnes réponses;
- 5% des élèves n'ont pas eu de bonne réponse; et
- la note moyenne de la classe était de 2,3 sur 3.



Détermine le pourcentage d'élèves qui ont eu exactement une bonne réponse ainsi que le pourcentage d'élève qui ont exactement 2 bonnes réponses.

Problem of the Week
Problem D and Solution
Percents of Interest

Problem

The following information is known about the results of a recent math test: there were three questions on the test, each question was worth 1 mark, each question was marked right or wrong (no part marks), 50% of the students got all 3 questions correct, 5% of the students got no questions correct, and the class average mark was 2.3 out of 3. Determine the percentage of students who got exactly 1 question correct and the percentage of students who got exactly 2 questions correct.

Solution

Solution 1

In this solution, we will use only one variable.

To determine an average, we must determine the total of all the scores and divide by the number of students. Without changing the overall class average, suppose that 100 students wrote this test.

Let x represent the percent who got exactly 2 questions correct.
Then $100 - 50 - 5 - x = (45 - x)$ percent got exactly 1 question correct.

Since 50% of the students got all 3 questions correct, 50 students each scored 3 marks and earned a total of $50 \times 3 = 150$ marks.

Since x% of the students got exactly 2 questions correct, x students each scored 2 marks and earned a total of $x \times 2 = 2x$ marks.

Since $(45 - x)$% of the students got exactly 1 question correct, $(45 - x)$ students each scored 1 mark and earned a total of $(45 - x) \times 1 = (45 - x)$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

The total number of marks earned by the 100 students was $150 + 2x + (45 - x) + 0 = x + 195$.

We know that the average score was 2.3, so

$$\begin{aligned} \frac{x + 195}{100} &= 2.3 \\ x + 195 &= 230 \\ x &= 35 \\ 45 - x &= 10 \end{aligned}$$

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.

Solution 2

In this solution, we will use two variables.

To determine an average, we must determine the total of all the scores and divide by the number of students. Without changing the overall class average, suppose that 100 students wrote this test.

Let x represent the percent who got exactly 2 questions correct.
Let y represent the percent who got exactly 1 question correct.

Then, $50 + x + y + 5 = 100$ which simplifies to $x + y = 45$. (1)

Since 50% of the students got all 3 questions correct, 50 students each scored 3 marks and earned a total of $50 \times 3 = 150$ marks.

Since x% of the students got exactly 2 questions correct, x students each scored 2 marks and earned a total of $x \times 2 = 2x$ marks.

Since y% of the students got exactly 1 question correct, y students each scored 1 mark and earned a total of $y \times 1 = y$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

The total number of marks earned by the 100 students was $150 + 2x + y + 0 = 2x + y + 150$.

We know that the average score was 2.3, so

$$\begin{aligned} \frac{2x + y + 150}{100} &= 2.3 \\ 2x + y + 150 &= 230 \\ 2x + y &= 80 \quad (2) \end{aligned}$$

Subtracting equation (1) from equation (2), we obtain $x = 35$. Substituting $x = 35$ into equation (1), we obtain $y = 10$.

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.

Solution 3

In this solution, we will use 2 variables but we will not assume a class size.

To determine an average, we must determine the total of all the scores and divide by the number of students.

Let n represent the number of students who wrote the test where n is a positive integer.

Let x represent the percent who got exactly 2 questions correct.
Let y represent the percent who got exactly 1 question correct.

Then, $50 + x + y + 5 = 100$ which simplifies to $x + y = 45$. (1)

Since 50% of the students got all 3 questions correct, $\frac{50}{100}n$ students each scored 3 marks and earned a total of $\frac{50}{100}n \times 3 = \frac{150n}{100}$ marks.

Since x% of the students got exactly 2 questions correct, $\frac{x}{100}n$ students each scored 2 marks and earned a total of $\frac{x}{100}n \times 2 = \frac{2xn}{100}$ marks.

Since y% of the students got exactly 1 question correct, $\frac{y}{100}n$ students each scored 1 mark and earned a total of $\frac{y}{100}n \times 1 = \frac{yn}{100}$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

The total number of marks earned by the n students was $\frac{150n}{100} + \frac{2xn}{100} + \frac{yn}{100} = \frac{n}{100}(2x + y + 150)$.

We know that the average score was 2.3 and n is a positive integer, so

$$\begin{aligned} \frac{\frac{n}{100}(2x + y + 150)}{n} &= 2.3 \\ 2x + y + 150 &= 230 \\ 2x + y &= 80 \quad (2) \end{aligned}$$

Subtracting equation (1) from equation (2), we obtain $x = 35$. Substituting $x = 35$ into equation (1), we obtain $y = 10$.

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.

Problème de la semaine

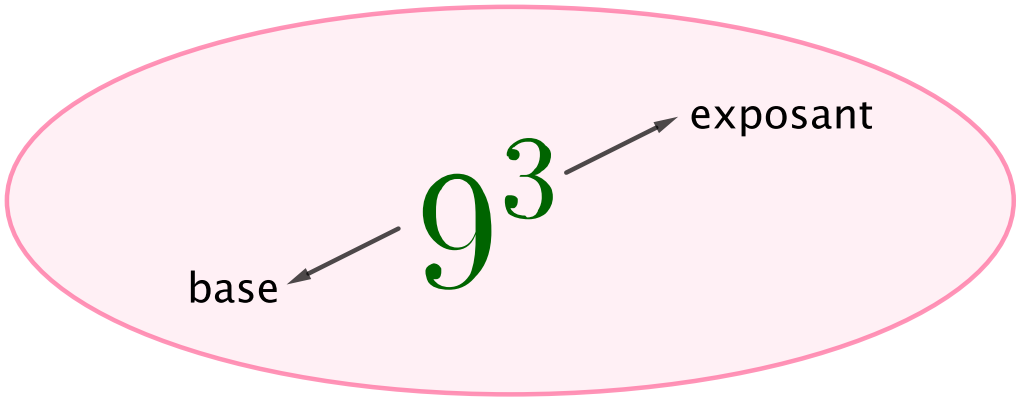
Problème D

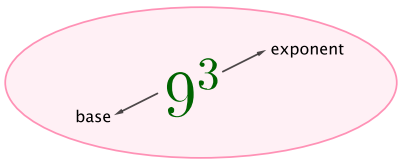
Jeu de puissance

Souvent, on peut écrire une puissance sous forme d'une autre puissance. Par exemple, $9^3 = 27^2$. Un autre exemple est $(-5)^4 = 25^2$.

Si x et y sont des nombres entiers, détermine toutes les paires (x, y) qui font que l'équation suivante est vraie.

$$(x-1)^{x+y} = 2^6.$$





Problem of the Week
Problem D and Solution
Power Play

Problem

Often a power can be rewritten as another power. For example $9^3 = 27^2$ or $(-5)^4 = 25^2$.

If x and y are integers, find all ordered pairs (x, y) that satisfy the equation $(x-1)^{x+y} = 2^6$.

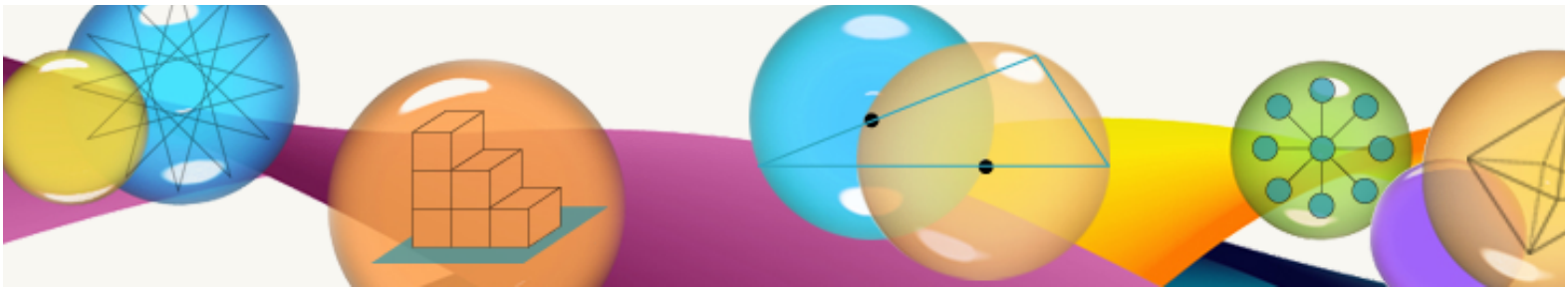
Solution

The number 2^6 equals 64.
We can express 64 as a^b where a and b are integers in the following ways:
$64^1, 8^2, 4^3, 2^6, (-2)^6,$ and $(-8)^2$.

We can use these powers and the expression $(x-1)^{x+y}$ to find values for x and y.

Power	$x-1$	$x+y$	x	y
64^1	64	1	65	-64
8^2	8	2	9	-7
4^3	4	3	5	-2
2^6	2	6	3	3
$(-2)^6$	-2	6	-1	7
$(-8)^2$	-8	2	-7	9

The six ordered pairs are $(65, -64), (9, -7), (5, -2), (3, 3), (-1, 7)$, and $(-7, 9)$.



Relations et équations

Amène-moi à la couverture

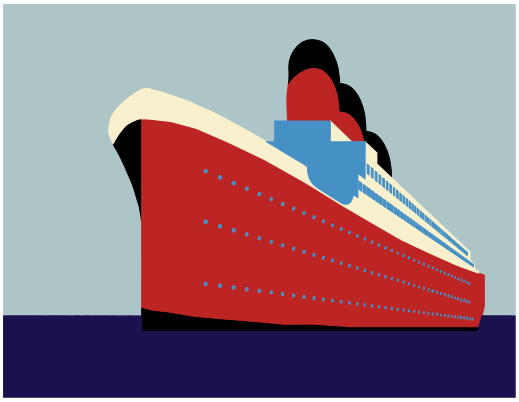
Problème de la semaine

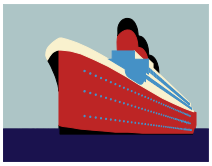
Problème D

Tout le monde à bord!

Un passager en croisière en Alaska a demandé au capitaine combien de passagers étaient à bord du paquebot. Le capitaine a répondu : $\frac{1}{6}$ des passagers sont des personnes âgées, $\frac{1}{4}$ des passagers sont des enfants ou des adolescents, il y a trois fois plus d'adultes que d'adolescents, et il y a 138 enfants à bord.

Combien y a-t-il de passagers sur le paquebot de croisière?





Problem of the Week
Problem D and Solution
All Aboard!

Problem

The captain of an Alaskan cruise was asked by one of the passengers how many guests were on board. The captain replied: $\frac{1}{6}$ of our guests are seniors, $\frac{1}{4}$ of our guests are children or teenagers, there are three times as many adults as teenagers, and there are 138 children on board. How many passengers are on board the cruise ship?

Solution

Let n be the total number of passengers.
Let s be the number of seniors, a be the number of adults, t be the number of teenagers, and c be the number of children.

Therefore, $n = s + a + t + c$.

Since $\frac{1}{6}$ of the passengers are seniors, $s = \frac{1}{6}n$.
Since $\frac{1}{4}$ of the passengers are children or teenagers, $c + t = \frac{1}{4}n$.
It is given that $c = 138$. Therefore, this becomes $138 + t = \frac{1}{4}n$, or $t = \frac{1}{4}n - 138$.
Since there are three times as many adults as teenagers, $a = 3t = 3(\frac{1}{4}n - 138)$.

Substituting into $n = s + a + t + c$, we have

$$\begin{aligned}
n &= \left(\frac{1}{6}n\right) + 3\left(\frac{1}{4}n - 138\right) + \left(\frac{1}{4}n - 138\right) + 138 \\
n &= \frac{1}{6}n + \frac{3}{4}n - 414 + \frac{1}{4}n - 138 + 138 \\
n &= \frac{1}{6}n + n - 414 \\
n &= \frac{7}{6}n - 414
\end{aligned}$$

Rearranging, we have $\frac{7}{6}n - n = 414$, and so $\frac{1}{6}n = 414$ or $n = 2484$.
Therefore, there are 2484 passengers on board the cruise ship.

Although not required, we can determine the number of adults is 1449, the number of seniors is 414, the number of teenagers is 483. We can use this to verify the given information.

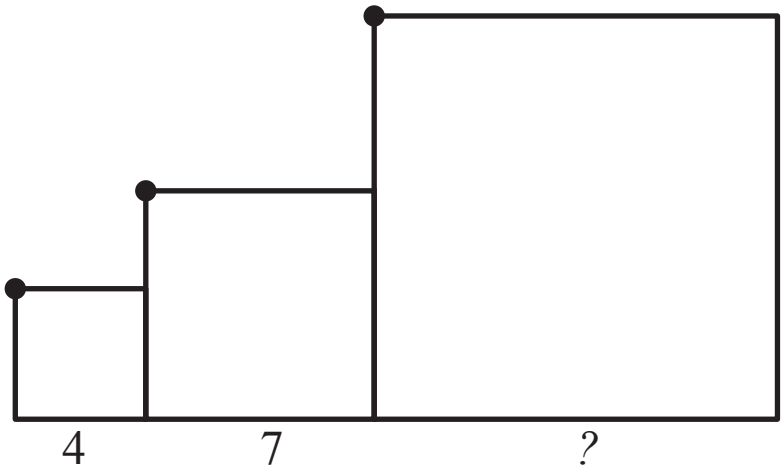
Problème de la semaine

Problème D

Grand, très grand, plus grand

Trois carrés sont placés l'un à côté de l'autre, comme le démontre le diagramme. Le côté du plus petit carré mesure 4 unités, celui du deuxième carré mesure 7 unités, mais la longueur du côté du plus grand carré est inconnue. Cependant, le coin supérieur gauche de chaque carré se trouve sur une même ligne droite.

Détermine la longueur du côté du plus grand carré.



Problem of the Week

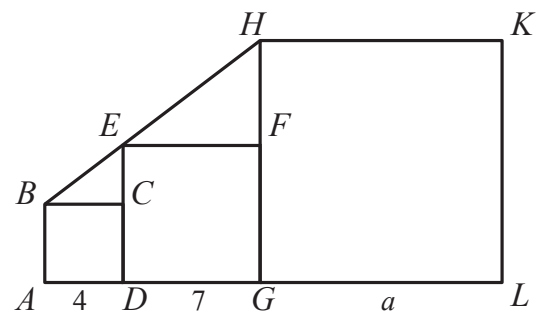
Problem D and Solution

Big, Bigger, Biggest

Problem

Three squares are placed beside each other as shown. The smallest square has side length 4 units, the middle-sized square has side length 7 units, but the side length of the largest square is unknown. However, the top left corner of each of the three squares lies on a straight line. Determine the side length of the largest square.

Solution



Label the vertices as shown on the diagram. Draw line segment BH through E. Let a represent the side length of the larger square.

In Solution 1, we will solve the problem by calculating the slope of BH.

In Solution 2, we will solve the problem using similar triangles.

In Solution 3, we will place the diagram on the xy-plane and solve the problem using analytic geometry.

Solution 1

The slope of a line is equal to its rise divided by its run.

If we look at the line segment from B to E, $BC = 4$ and $CE = DE - DC = 7 - 4 = 3$. Therefore, slope $BE = \frac{CE}{BC} = \frac{3}{4}$.

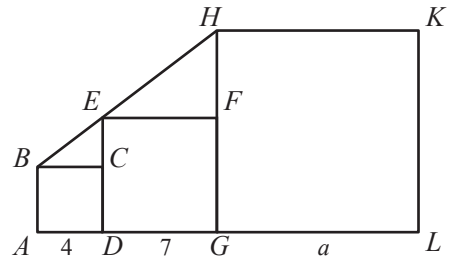
If we look at the line segment from E to H, $EF = 7$ and $FH = GH - GF = a - 7$. Therefore, slope $EH = \frac{FH}{EF} = \frac{a-7}{7}$.

Since B, E and H lie on a straight line, slope BE must equal slope EH. Therefore,

$$\begin{aligned} \text{slope } BE &= \text{slope } EH \\ \frac{3}{4} &= \frac{a-7}{7} \\ 4(a-7) &= 3(7) \\ 4a - 28 &= 21 \\ 4a &= 49 \end{aligned}$$

$\therefore a = \dfrac{49}{4}$ and the side length of the larger square is $\dfrac{49}{4}$ units.

Solution 2



Consider $\triangle BCE$ and $\triangle EFH$. We will first show that $\triangle BCE \sim \triangle EFH$.

Since $ABCD$ is a square, $\angle BCD = 90^\circ$.
Therefore, $\angle BCE = 180^\circ - \angle BCD = 180^\circ - 90^\circ = 90^\circ$.

Since $DEFG$ is a square, $\angle EFG = 90^\circ$.
Therefore, $\angle EFH = 180^\circ - \angle EFG = 180^\circ - 90^\circ = 90^\circ$.
Thus, $\angle BCE = \angle EFH$.

Since $ABCD$ and $DEFG$ are squares and AG is a straight line, BC is parallel to EF.
Therefore, $\angle EBC$ and $\angle HEF$ are corresponding angles and so $\angle EBC = \angle HEF$.

Since the angles in a triangle add to 180°, then we must also have $\angle BEC = \angle EHF$.

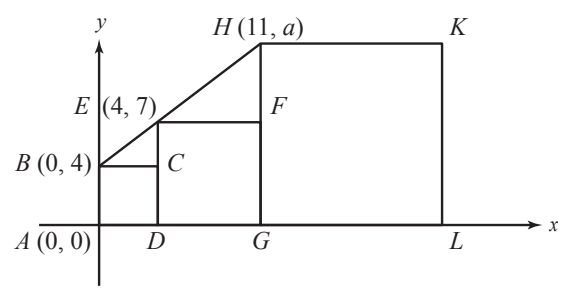
Therefore, $\triangle BCE \sim \triangle EFH$, by Angle−Angle−Angle Triangle Similarity.

Since $\triangle BCE \sim \triangle EFH$, corresponding side lengths are in the same ratio. In particular,

$$\begin{aligned}
\frac{EC}{BC} &= \frac{HF}{EF} \\
\frac{DE - DC}{BC} &= \frac{GH - GF}{EF} \\
\frac{7-4}{4} &= \frac{a-7}{7} \\
\frac{3}{4} &= \frac{a-7}{7} \\
4(a-7) &= 3(7) \\
4a - 28 &= 21 \\
4a &= 49
\end{aligned}$$

$\therefore a = \dfrac{49}{4}$ and the side length of the larger square is $\dfrac{49}{4}$ units.

Solution 3



We proceed by placing the diagram on the xy-plane with A at $(0,0)$ and AL along the x-axis.
The coordinates of B are $(0,4)$, the coordinates of E are $(4,7)$, and the coordinates of H are $(11,a)$.
Let's determine the equation of the line through B, E, H.
Since this line passes through $(0,4)$, it has y-intercept 4.
Since the line passes through $(0,4)$ and $(4,7)$, it has slope $= \frac{7-4}{4-0} = \frac{3}{4}$.
Therefore, the equation of the line is $y = \frac{3}{4}x + 4$.

Since $H(11,a)$ lies on this line, substituting $x = 11$ and $y = a$ into $y = \frac{3}{4}x + 4$, we have

$$a = \frac{3}{4}(11) + 4 = \frac{33}{4} + 4 = \frac{33+16}{4} = \frac{49}{4}$$

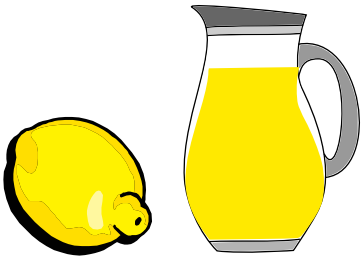
$\therefore$ the side length of the larger square is $\dfrac{49}{4}$ units.

Problème de la semaine

Problème D

Concentré

Un contenant de jus contient un mélange de concentré de jus de citron et d'eau. Lorsqu'un litre d'eau est ajouté au contenant, le rapport entre la masse du concentré de jus et celle de l'eau est de 1 : 2. Lorsqu'un litre de concentré de jus est ajouté à ce nouveau mélange, le rapport devient de 2 : 3. Détermine le rapport entre la masse du concentré de jus et celle de l'eau, dans le mélange original.



Problem of the Week
Problem D and Solution
Concentrate

Problem

A large jug of lemonade contains a mixture of lemon juice concentrate and water. When 1 litre of water is added to the jug, the ratio, by volume, of juice concentrate to water is $1:2$. When 1 litre of juice concentrate is added to the new mixture, the ratio becomes $2:3$. Find the original ratio, by volume, of lemon juice concentrate to water in the jug.

Solution

Let j be the amount of juice, in litres, in the original mixture.
Let w be the amount of water, in litres, in the original mixture.

When 1 litre of water is added, the ratio of juice concentrate to water is $1:2$. This tells us that

$$\frac{j}{w+1} = \frac{1}{2} \qquad (1)$$

Simplifying (1), we obtain $w+1=2j$ and $w=2j-1$ follows. (2)

When 1 litre of juice concentrate is added to the new mixture, the ratio becomes $2:3$. This tells us that

$$\frac{j+1}{w+1} = \frac{2}{3} \qquad (3)$$

From (2), since $w=2j-1$, we can substitute for w in (3), obtaining:

$$\begin{aligned}
\frac{j+1}{2j-1+1} &= \frac{2}{3} \\
\frac{j+1}{2j} &= \frac{2}{3} \\
2(2j) &= 3(j+1) \\
4j &= 3j+3 \\
j &= 3
\end{aligned}$$

Substituting $j=3$ in $w+1=2j$ we obtain $w+1=6$ and $w=5$ follows.

So there are 3 litres of juice concentrate in the container and 5 litres of water. The ratio of juice concentrate to water is $3:5$.

Problème de la semaine

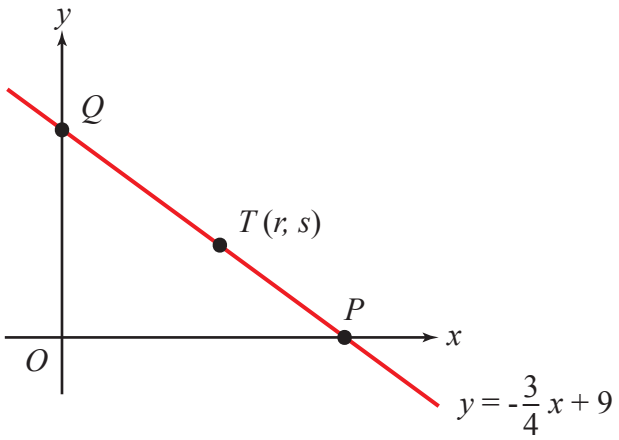
Problème D

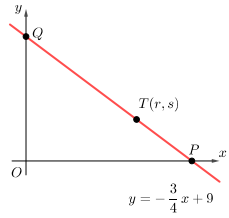
Le point de division

La droite $y = -\frac{3}{4}x + 9$ croise l'axe des x au point P et l'axe des y au point Q.

Le point $T(r, s)$ se situe sur le segment PQ de façon à ce que l'aire de POQ soit trois fois celle de TOP.

Détermine les valeurs de r et s, les coordonnées du point T.





Problem of the Week

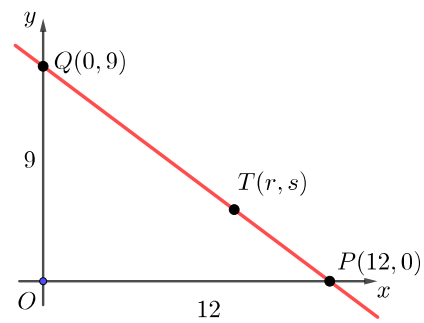
Problem D and Solution

A Point of Division

Problem

The line $y = -\frac{3}{4}x + 9$ crosses the x-axis at P and the y-axis at Q. Point $T(r, s)$ lies on the line segment PQ such that the area of $\triangle POQ$ is three times the area of $\triangle TOP$. Determine the values of r and s, the coordinates of T.

Solution

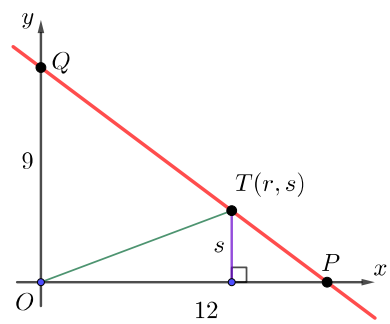


We begin by calculating the coordinates of P and Q, the x- and y-intercepts of the line $y = -\frac{3}{4}x + 9$.

Since the equation of the line is written in the form $y = mx + b$ where b is the y-intercept of the line, the y-intercept is 9 and so the coordinates of Q are $(0, 9)$. To determine the x-intercept, set $y = 0$ to obtain $0 = -\frac{3}{4}x + 9 \implies \frac{3}{4}x = 9 \implies x = 12$. Thus, P has coordinates $(12, 0)$.

We now present two different solutions to the problem.

Solution 1



Since $\triangle POQ$ is a right triangle with base $PO = 12$ and height $OQ = 9$, using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have $\text{area}(\triangle POQ) = \frac{12 \times 9}{2} = 54$.

Since the area of $\triangle POQ$ is three times the area of $\triangle TOP$, $\text{area}(\triangle TOP) = \frac{1}{3}(\text{area}(\triangle POQ)) = \frac{1}{3}(54) = 18$.

$\triangle TOP$ has area 18, base $PO = 12$ and height s. Using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have

$$\begin{aligned} \text{area}(\triangle TOP) &= \frac{PO \times s}{2} \\ 18 &= \frac{12 \times s}{2} \\ 18 &= 6s \\ \therefore\ s &= 3 \end{aligned}$$

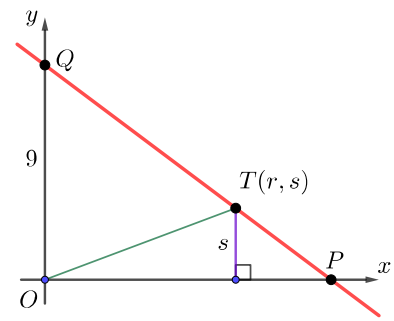
$T(r, s)$ lies on the line $y = -\frac{3}{4}x + 9$ and $s = 3$ so we can substitue $x = r$ and $y = 3$

$$\begin{aligned} 3 &= -\frac{3}{4}r + 9 \\ \frac{3}{4}r &= 6 \\ \therefore\ r &= 8 \end{aligned}$$

Therefore, T is the point $(r, s) = (8, 3)$.

Solution 2

If two triangles have equal bases, the areas of the triangles are proportional to the heights of the triangles.



$\triangle POQ$ and $\triangle TOP$ have the same base, OP.

Since the area of $\triangle POQ$ is 3 times the area of $\triangle TOP$, then the height of $\triangle POQ$ is 3 times the height of $\triangle TOP$. In other words, the height of $\triangle TOP$ is $\frac{1}{3}$ the height of $\triangle POQ$. $\triangle POQ$ has height $OQ = 9$ and $\triangle TOP$ has height s. Therefore, $s = \frac{1}{3}(OQ) = \frac{1}{3}(9) = 3$.

Since $T(r, s)$ lies on the line $y = -\frac{3}{4}x + 9$, we have

$$\begin{aligned} s &= -\frac{3}{4}r + 9 \\ 3 &= -\frac{3}{4}r + 9 \\ \frac{3}{4}r &= 6 \\ \therefore\ r &= 8 \end{aligned}$$

Therefore, T is the point $(r, s) = (8, 3)$.

Note that it was actually unnecessary to find the x-intercept for the second solution as it was never used in the second solution.

For Further Thought:

Find the coordinates of S, another point on line segment QP, so that

$$\text{the area of } \triangle SOQ = \text{ the area of } \triangle TOP,$$

thus creating three triangles of equal area. How are the points Q, S, T, and P related?

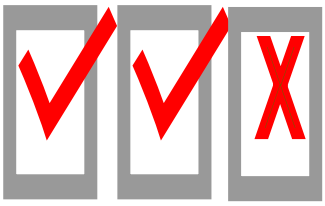
Problème de la semaine

Problème D

Pourcentage d'intérêt

L'information suivante est connue au sujet des résultats d'un test de mathématiques :

- le test possédait trois questions;
- chaque question avait une valeur de 1 point;
- chaque réponse était évaluée, soit correcte ou incorrecte;
- 50% des élèves ont eu les 3 bonnes réponses;
- 5% des élèves n'ont pas eu de bonne réponse; et
- la note moyenne de la classe était de 2,3 sur 3.



Détermine le pourcentage d'élèves qui ont eu exactement une bonne réponse ainsi que le pourcentage d'élève qui ont exactement 2 bonnes réponses.

Problem of the Week
Problem D and Solution
Percents of Interest

Problem

The following information is known about the results of a recent math test: there were three questions on the test, each question was worth 1 mark, each question was marked right or wrong (no part marks), 50% of the students got all 3 questions correct, 5% of the students got no questions correct, and the class average mark was 2.3 out of 3. Determine the percentage of students who got exactly 1 question correct and the percentage of students who got exactly 2 questions correct.

Solution

Solution 1

In this solution, we will use only one variable.

To determine an average, we must determine the total of all the scores and divide by the number of students. Without changing the overall class average, suppose that 100 students wrote this test.

Let x represent the percent who got exactly 2 questions correct.
Then $100 - 50 - 5 - x = (45 - x)$ percent got exactly 1 question correct.

Since 50% of the students got all 3 questions correct, 50 students each scored 3 marks and earned a total of $50 \times 3 = 150$ marks.

Since x% of the students got exactly 2 questions correct, x students each scored 2 marks and earned a total of $x \times 2 = 2x$ marks.

Since $(45 - x)$% of the students got exactly 1 question correct, $(45 - x)$ students each scored 1 mark and earned a total of $(45 - x) \times 1 = (45 - x)$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

The total number of marks earned by the 100 students was $150 + 2x + (45 - x) + 0 = x + 195$.

We know that the average score was 2.3, so

$$\begin{aligned} \frac{x + 195}{100} &= 2.3 \\ x + 195 &= 230 \\ x &= 35 \\ 45 - x &= 10 \end{aligned}$$

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.

Solution 2

In this solution, we will use two variables.

To determine an average, we must determine the total of all the scores and divide by the number of students. Without changing the overall class average, suppose that 100 students wrote this test.

Let x represent the percent who got exactly 2 questions correct.
Let y represent the percent who got exactly 1 question correct.

Then, $50 + x + y + 5 = 100$ which simplifies to $x + y = 45$. (1)

Since 50% of the students got all 3 questions correct, 50 students each scored 3 marks and earned a total of $50 \times 3 = 150$ marks.

Since x% of the students got exactly 2 questions correct, x students each scored 2 marks and earned a total of $x \times 2 = 2x$ marks.

Since y% of the students got exactly 1 question correct, y students each scored 1 mark and earned a total of $y \times 1 = y$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

The total number of marks earned by the 100 students was $150 + 2x + y + 0 = 2x + y + 150$.

We know that the average score was 2.3, so

$$\begin{aligned} \frac{2x + y + 150}{100} &= 2.3 \\ 2x + y + 150 &= 230 \\ 2x + y &= 80 \quad (2) \end{aligned}$$

Subtracting equation (1) from equation (2), we obtain $x = 35$. Substituting $x = 35$ into equation (1), we obtain $y = 10$.

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.

Solution 3

In this solution, we will use 2 variables but we will not assume a class size.

To determine an average, we must determine the total of all the scores and divide by the number of students.

Let n represent the number of students who wrote the test where n is a positive integer.

Let x represent the percent who got exactly 2 questions correct.
Let y represent the percent who got exactly 1 question correct.

Then, $50 + x + y + 5 = 100$ which simplifies to $x + y = 45$. (1)

Since 50% of the students got all 3 questions correct, $\frac{50}{100}n$ students each scored 3 marks and earned a total of $\frac{50}{100}n \times 3 = \frac{150n}{100}$ marks.

Since x% of the students got exactly 2 questions correct, $\frac{x}{100}n$ students each scored 2 marks and earned a total of $\frac{x}{100}n \times 2 = \frac{2xn}{100}$ marks.

Since y% of the students got exactly 1 question correct, $\frac{y}{100}n$ students each scored 1 mark and earned a total of $\frac{y}{100}n \times 1 = \frac{yn}{100}$ marks.

Since 5% of the students got no questions correct, 5 students scored 0 marks and earned a total of $5 \times 0 = 0$ marks.

The total number of marks earned by the n students was $\frac{150n}{100} + \frac{2xn}{100} + \frac{yn}{100} = \frac{n}{100}(2x + y + 150)$.

We know that the average score was 2.3 and n is a positive integer, so

$$\begin{aligned} \frac{\frac{n}{100}(2x + y + 150)}{n} &= 2.3 \\ 2x + y + 150 &= 230 \\ 2x + y &= 80 \quad (2) \end{aligned}$$

Subtracting equation (1) from equation (2), we obtain $x = 35$. Substituting $x = 35$ into equation (1), we obtain $y = 10$.

Therefore, 35% of the students got exactly 2 questions correct and 10% of the students got exactly 1 question correct.

Problème de la semaine

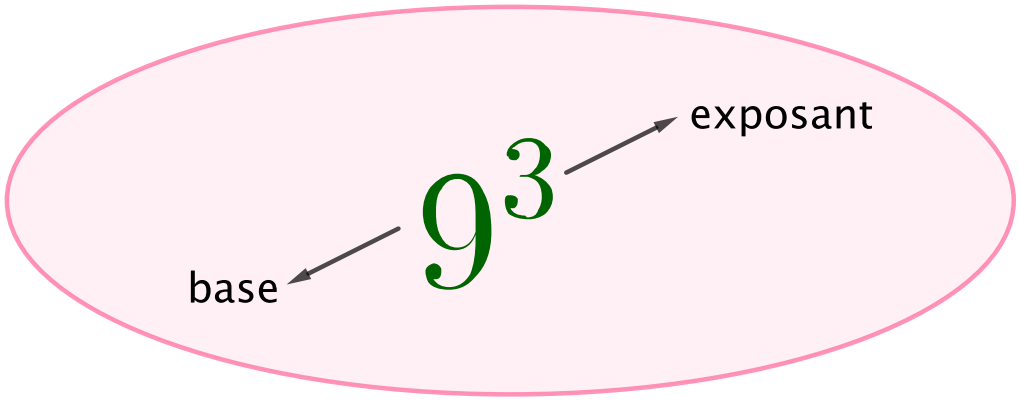
Problème D

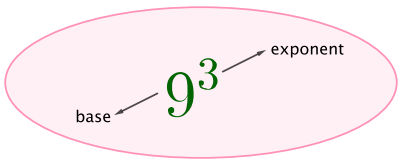
Jeu de puissance

Souvent, on peut écrire une puissance sous forme d'une autre puissance. Par exemple, $9^3 = 27^2$. Un autre exemple est $(-5)^4 = 25^2$.

Si x et y sont des nombres entiers, détermine toutes les paires (x, y) qui font que l'équation suivante est vraie.

$$(x-1)^{x+y} = 2^6.$$





Problem of the Week
Problem D and Solution
Power Play

Problem

Often a power can be rewritten as another power. For example $9^3 = 27^2$ or $(-5)^4 = 25^2$.

If x and y are integers, find all ordered pairs (x, y) that satisfy the equation $(x-1)^{x+y} = 2^6$.

Solution

The number 2^6 equals 64.
We can express 64 as a^b where a and b are integers in the following ways:
$64^1, 8^2, 4^3, 2^6, (-2)^6,$ and $(-8)^2$.

We can use these powers and the expression $(x-1)^{x+y}$ to find values for x and y.

Power	$x-1$	$x+y$	x	y
64^1	64	1	65	-64
8^2	8	2	9	-7
4^3	4	3	5	-2
2^6	2	6	3	3
$(-2)^6$	-2	6	-1	7
$(-8)^2$	-8	2	-7	9

The six ordered pairs are $(65, -64), (9, -7), (5, -2), (3, 3), (-1, 7)$, and $(-7, 9)$.