Problem of the Week
Grade 9/10 (D)
Problem and Solutions
2018 - 2019

Strands

Data Management & Probability
Measurement & Trigonometry
Number Sense & Algebra
Relations & Systems

(Click the strand name above to jump to that section)

The problems in this booklet are organized into strands. A problem often appears in multiple strands. The problems are suitable for most students in Grade 9 or higher.
Data Management & Probability
Problem of the Week
Problem D
New Heights

The heights of the ten members of the Riky’s basketball team were measured. The heights, in cm, of his nine teammates are 180, 181, 183, 187, 188, 190, 193, 195, and 196. Riky’s height is also a whole number.

The coach of the team made a mistake when measuring Riky’s height. After this mistake was corrected, both the mean and median of the heights increased by 0.5 cm.

Find all possible correct values for Riky’s height.

The mean of a list of numbers is the sum of the numbers in the list divided by the number of numbers in the list.

The median of a list of numbers is the middle number in the ordered list of numbers when there is an odd number of numbers in the list. The median of a list of numbers is the average of the two middle numbers in the ordered list of numbers when there is an even number of numbers in the list.
Problem of the Week
Problem D and Solution
New Heights

Problem
The heights of the ten members of Riky’s basketball team were measured. The heights, in cm, of his nine teammates are 180, 181, 183, 187, 188, 190, 193, 195, and 196. Riky’s height is also a whole number. The coach of the team made a mistake when measuring Riky’s height. After this mistake was corrected, both the mean and median of the heights increased by 0.5 cm. Find all possible correct values for Riky’s height.

Solution
We will let Riky’s correct height be \( R \) and the incorrectly recorded height be \( r \).
To find a relationship between \( R \) and \( r \) we will look at the averages of the heights. The average of the heights with the incorrect height, \( r \), is
\[
\frac{180+181+183+187+188+190+193+195+196+r}{10} = \frac{1693+r}{10}.
\]
The average with the correct height, \( R \), is
\[
\frac{180+181+183+187+188+190+193+195+196+R}{10} = \frac{1693+R}{10}.
\]
Therefore,
\[
\frac{1693 + r}{10} + 0.5 = \frac{1693 + R}{10}
\]
\[
1693 + r + 5 = 1693 + R
\]
\[
r + 5 = R
\]
Let’s look at the median for different values of \( r \).
Case 1: If \( r \leq 182 \), then \( R \leq 187 \) (since \( R = r + 5 \)).
The median with the incorrect height is \( \frac{187+188}{2} = 187.5 \). Since \( R \) and \( r \) are both less than 188, the median with the correct height will remain at 187.5. Therefore, this case is not a possible solution.
Case 2: If \( r = 183 \), then \( R = 188 \).
The median with the incorrect height is \( \frac{187+188}{2} = 187.5 \). Since \( R = 188 \), the median with the correct height is \( \frac{188+188}{2} = 188 \), which is an increase of 0.5 in the median. Therefore, this case is one possible solution.
Case 3: If \( r = 184 \), then \( R = 189 \).
The median with the incorrect height is \( \frac{187+188}{2} = 187.5 \). Since \( R = 189 \), the median with the correct height is \( \frac{188+189}{2} = 188.5 \), which is an increase of 1 in the median. Therefore, this case is not a possible solution.
Case 4: If $185 \leq r \leq 187$, then $190 \leq R \leq 192$.
The median with the incorrect height is $\frac{187+188}{2} = 187.5$. Since $190 \leq R \leq 193$, the median with the correct height is $\frac{188+190}{2} = 189$, which is an increase of 1.5 in the median. Therefore, this case is not a possible solution.

Case 5: If $r = 188$, then $R = 193$.
The median with the incorrect height is $\frac{188+188}{2} = 188$. Since $R = 193$, the median with the correct height is $\frac{188+190}{2} = 189$, which is an increase of 1 in the median. Therefore, this case is not a possible solution.

Case 6: If $r = 189$, then $R = 194$.
The median with the incorrect height is $\frac{188+189}{2} = 188.5$. Since $R = 194$, the median with the correct height is $\frac{188+190}{2} = 189$, which is an increase of 0.5 in the median. Therefore, this case is one possible solution.

Case 7: If $r \geq 190$, then $R \geq 195$.
The median with the incorrect height is $\frac{188+190}{2} = 189$. Since both $r$ and $R$ are greater than or equal to 190 the median with the correct height will remain at $\frac{188+190}{2} = 189$. Therefore, this case is not a possible solution.

Therefore, the possible correct heights for Riky are 188 cm or 194 cm.
Problem of the Week
Problem D
Different Dice, Same Outcome

When a mathematician says ‘a fair die’, they mean there is an equally likely chance of landing on any face of the die.

A standard six-sided die has its faces marked with the numbers 1, 2, 3, 4, 5, and 6. The die is fair and each number is used exactly once.

A special six-sided die has its faces marked with the numbers 1, 3, 4, 5, 6, and 8. The die is fair and each number is used exactly once.

Is it possible to create a second fair, six-sided die marked so that this die and the first special die can be used together to play a board game like Monopoly? (Numbers on the faces of this new die may appear more than once.)

In other words, does another fair, six-sided die exist so that when this new die and the special die are thrown the sum of the numbers on the top faces of the two dice range from 2 to 12 and the probability of obtaining each sum is the same as it would be if two standard dice had been thrown? If it is possible, what numbers would be on the faces of this new die? If it is not possible, explain why not.
Problem of the Week
Problem D and Solution
Different Dice, Same Outcome

Problem
For mathematicians, a fair die means there is an equally likely chance of landing on any face of the
die. A standard six-sided die has its faces marked with the numbers 1, 2, 3, 4, 5, and 6. The die is fair
and each number is used exactly once. A special six-sided die has its faces marked with the numbers 1,
3, 4, 5, 6, and 8. The die is fair and each number is used exactly once. Is it possible to create a second
fair, six-sided die marked so that this die and the first special die can be used together to play a board
game like Monopoly? (Numbers on the faces of this new die may appear more than once.) In other
words, does another fair, six-sided die exist so that when this new die and the special die are thrown
the sum of the numbers on the top faces of the two dice range from 2 to 12 and the probability of
obtaining each sum is the same as it would be if two standard dice had been thrown? If it is possible,
what numbers would be on the faces of this new die? If it is not possible, explain why not.

Solution
We first examine what happens when two standard dice are thrown. For each possible roll, we
calculate the sum of numbers on the top faces of the dice, and compile this information in a
table.

<table>
<thead>
<tr>
<th>Die 1</th>
<th>Die 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>3</td>
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<td>3</td>
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<td>5</td>
<td>6</td>
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<tr>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

From this table we can determine the probability of each sum by counting the number of ways
to get that sum and dividing by 36, the size of the sample space.

\[
\begin{align*}
P(\text{total} = 2) = P(\text{total} = 12) &= \frac{1}{36} \\
P(\text{total} = 4) = P(\text{total} = 10) &= \frac{3}{36} \\
P(\text{total} = 6) = P(\text{total} = 8) &= \frac{5}{36} \\
\end{align*}
\]

We need our new die and the special die to give these same probabilities when rolled together.
Since both the special and the new die have 6 sides, the size of the sample space will still be
6 \times 6 = 36.

We need the sums of the top faces to range from 2 to 12. Since the numbers on our special die
are 1, 3, 4, 5, 6 and 8, this means that on our new die, the smallest number must be 1 and the
largest number must be 4. Now, let’s examine the particular probabilities.

We need the probability that a sum of 2 is thrown to be \(\frac{1}{36}\). Since the size of the sample space
is 36, this means that we need exactly 1 way to get a sum of 2. Therefore, exactly one face on
our new die must have the number 1 on it.
We need the probability that a sum of 12 is thrown to be $\frac{1}{36}$. Again, since the size of the sample space is 36, this means that we need exactly 1 way to get a sum of 12. Therefore, exactly one face on the new die must have the number 4 on it. Therefore, on the new die the numbers on the six faces range from 1 to 4, and exactly one face has a 1 on it and exactly one face has a 4 on it. Therefore, there must be only 2’s and 3’s on the remaining 4 faces.

We need the probability that a sum of 3 is thrown to be $\frac{2}{36}$. Since the size of the sample space is 36, this means that there must be exactly 2 ways to get a sum of 3. If our new die has one face with a 2 on it, then there would be one way to get a sum of three: by throwing a 1 on the special die and a 2 on our new die. Therefore, we would need a second 2 on the new die. If there was third face with a 2, then there would be a third way to roll a sum of 3, which is too many.

Similarly, we need the probability that a sum of 11 is thrown to be $\frac{2}{36}$. Since the size of the sample space is 36, this means that there must be exactly 2 ways to get a sum of 11. If our new die has one face with a 3 on it, then there would be one way to get a sum of eleven: by throwing an 8 on the special die and a 3 on our new die. Therefore, we would need a second 3 on the new die. If there was a third face with a 3, then there would be a third way to roll a sum of 11, which is too many. Therefore, the faces on the new die, if it exists, will be 1, 2, 2, 3, 3, 4.

Let’s check that this new die satisfies the conditions of the problem. For each possible roll with the special die and this new die, we calculate the sum of the numbers on the top faces of the dice and compile this information in a table.

<table>
<thead>
<tr>
<th>Special Die</th>
<th>New Die</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>3</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>10</td>
<td></td>
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<tr>
<td>8</td>
<td>9</td>
<td>10</td>
<td>10</td>
<td>11</td>
<td>11</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

From this table, we can determine the probability of different sums. For each possible sum, we count the number of ways to get that sum, and divide by 36, the size of the sample space.

- $P(\text{total } = 2) = P(\text{total } = 12) = \frac{1}{36}$
- $P(\text{total } = 3) = P(\text{total } = 11) = \frac{2}{36}$
- $P(\text{total } = 4) = P(\text{total } = 10) = \frac{3}{36}$
- $P(\text{total } = 5) = P(\text{total } = 9) = \frac{4}{36}$
- $P(\text{total } = 6) = P(\text{total } = 8) = \frac{5}{36}$
- $P(\text{total } = 7) = \frac{6}{36}$

Therefore, a second die exists with faces numbered 1, 2, 2, 3, 3, and 4, such that when this die and the special die are thrown, the probability of obtaining each sum is the same as it would be if two standard dice had been thrown. The existence of this die may be surprising. Can you find another pair of special dice that can do the same thing?
A box contains a total of 400 tickets that come in five colours: blue, green, red, yellow and orange. The ratio of blue to green to red tickets is $1 : 2 : 4$. The ratio of green to yellow to orange tickets is $1 : 3 : 6$. How many tickets are there of each colour?
Problem of the Week
Problem D and Solution
Get Your Tickets

Problem
A box contains a total of 400 tickets that come in five colours: blue, green, red, yellow and orange. The ratio of blue to green to red tickets is $1 : 2 : 4$. The ratio of green to yellow to orange tickets is $1 : 3 : 6$. How many tickets are there of each colour?

Solution
Solution 1
We denote the number of tickets of each of the five colours by the first letter of the colour. We are given that $b : g : r = 1 : 2 : 4$ and that $g : y : o = 1 : 3 : 6$. Through multiplication by 2, the ratio $1 : 3 : 6$ is equivalent to the ratio $2 : 6 : 12$. Thus, $g : y : o = 2 : 6 : 12$.
We chose to scale this ratio by a factor of 2 so that the only colour common to the two given ratios, green, now has the same number in both of these ratios.
That is, $b : g : r = 1 : 2 : 4$ and $g : y : o = 2 : 6 : 12$ and since the term $g$ is 2 in each ratio, then we can combine these to form a single ratio,

$$b : g : r : y : o = 1 : 2 : 4 : 6 : 12$$

This ratio tells us that for every blue ticket, there are 2 green, 4 red, 6 yellow, and 12 orange tickets. Thus, if there was only 1 blue ticket, then there would be $1 + 2 + 4 + 6 + 12 = 25$ tickets in total.
However, we are given that the box contains 400 tickets in total. Therefore, the number of blue tickets in the box is $\frac{400}{25} = 16$.
Therefore, there are 16 blue, 32 green, 64 red, 96 yellow, and 192 orange tickets. (Note that there are $16 + 32 + 64 + 96 + 192 = 400$ tickets in total.)
Solution 2
We denote the number of tickets of each of the five colours by the first letter of the colour. We are given that $b : g : r = 1 : 2 : 4$ and that $g : y : o = 1 : 3 : 6$.

The first ratio tells us that $\frac{b}{g} = \frac{1}{2}$, and so $b = \frac{g}{2}$.

The first ratio also tells us that $\frac{g}{r} = \frac{2}{4}$, and so $r = 2g$.

The second ratio tells us that $\frac{g}{y} = \frac{1}{3}$, and so $y = 3g$.

The second ratio also tells us that $\frac{g}{o} = \frac{1}{6}$, and so $o = 6g$.

We are given that there are a total of 400 tickets. That is, $b + g + r + y + o = 400$.
Substituting $b = \frac{g}{2}, r = 2g, y = 3g,$ and $o = 6g,$ this becomes

$$\frac{g}{2} + g + 2g + 3g + 6g = 400$$

$$\frac{25}{2}g = 400$$

$$g = 32$$

Thus, $b = \frac{g}{2} = 16, \ r = 2g = 64, \ y = 3g = 96,$ and $o = 6g = 192$.

Therefore, there are 16 blue, 32 green, 64 red, 96 yellow, and 192 orange tickets.
Problem of the Week
Problem D
Marble Maze

In MATHO, the game described below, what is the probability of winning?

A marble is dropped into the maze at S. The marble will move down through the maze in the direction of the arrows. At any junction, the marble is equally likely to go in any of the possible downward paths. If the marble ends up in A, you lose. Otherwise, you win.

A Probability Note:

On a fair, six-sided die, with sides labelled 1 to 6, the probability of rolling each of the numbers from 1 to 6 is equally likely. So the probability of rolling a 3 is \( \frac{1}{6} \).

If you pick up that die and roll it again, the probability of rolling a 4 is \( \frac{1}{6} \).

The probability of rolling a 3 and then rolling a 4 is therefore \( \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \).
Problem of the Week
Problem D and Solution
Marble Maze

Problem

In MATHO, the game described below, what is the probability of winning?
In the game MATHO, a marble is dropped into the maze at S. The marble will move down through the maze in the direction of the arrows. At any junction, the marble is equally likely to go in any of the possible downward paths. If the marble ends up in A, you lose. Otherwise, you win.

Solution

We label the five junctions as V, W, X, Y, and Z.

We will calculate the probability of losing MATHO. The probability of winning can then be obtained by subtracting the probability of losing from 1. You lose the game if your marble ends up at A. From the arrows the marble can follow, we see that in order to get to A, the marble must go through X (from X the marble must go to A). So we will calculate the probability that the marble goes to X.

To get to X, the marble can go from S to V to W to X, or S to V to X, or S to V to Y to X.

At V, the probability that the marble goes down any of the three paths (that is, towards W, X or Y) is $\frac{1}{3}$. So the probability that the marble goes directly from V to X is $\frac{1}{3}$.

At W, the probability that the marble turns to X is $\frac{1}{2}$, so the probability that the marble goes from V to W and from W to X is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

At Y, the probability that the marble turns to X is $\frac{1}{3}$, so the probability that the marble goes from V to Y and Y to X is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

Therefore, the probability that the marble gets to X (and thus ends up at A) is $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$. That is, the probability of losing the game is $\frac{11}{18}$. Therefore, the probability of winning the game is $1 - \frac{11}{18} = \frac{7}{18}$, or approximately 38.9%.

Extension: A fair game is a game in which there is an equal chance of winning or losing. If a game is fair, then we can say that the probability of winning is equal to the probability of losing. MATHO is not a fair game. Can you create a modified version of MATHO that is fair?

A Final Thought: Notice that there are 3 different paths from S to A (SVWXA, SVXA, SVYXA) and the total number of possible paths that the marble could take is 7 (check this for yourself!). You might be tempted to conclude that the probability of the marble ending up at A is $\frac{3}{7}$. This is not the case. You would be incorrectly assuming that the marble is equally likely to take each path. We are given that at any junction, the marble is equally likely to go in any of the downward paths. We are not told that each path is equally likely. In fact, each path is not equally likely, which is why we cannot determine the probability this way.
Measurement
&
Trigonometry

TATE ME TO THE COVER
Problem of the Week
Problem D
Dividing a Square

In the diagram below, the large square $ABCD$ is divided into three regions, a smaller square $BFGH$ and two rectangles $AHGE$ and $CDEF$. The area of square $BFGH$ is $36$ cm$^2$.

We are also given that:

$$\text{Area of } BFGH - \text{Area of } CDEF = \text{Area of } CDEF - \text{Area of } AHGE$$

Determine the areas of rectangle $CDEF$ and rectangle $AHGE$. 
Problem
The large square $ABCD$ (shown above) is divided into three regions, a smaller square $BFGH$ and two rectangles $AHGE$ and $CDEF$. The area of square $BFGH$ is 36 cm$^2$.

We are also given that:
Area of $BFGH$ − Area of $CDEF$ = Area of $CDEF$ − Area of $AHGE$

Determine the areas of rectangle $CDEF$ and rectangle $AHGE$.

Solution
Solution 1

We will use the diagram to the right in our solution.

Let the side length of $ABCD$ be $s$ cm.

Since the area of $BFGH = 36$ cm$^2$, then $HB = BF = 6$ cm.

Therefore, $AE = BF = 6$ cm and $AH = AB − HB = (s − 6)$ cm.

Similarly, $ED = (s − 6)$ cm.

Using the given equation,

$$\text{Area of } BFGH − \text{Area of } CDEF = \text{Area of } CDEF − \text{Area of } AHGE$$

$$36 − s(s − 6) = s(s − 6) − 6(s − 6)$$

$$36 − s^2 + 6s = s^2 − 6s − 6s + 36$$

$$18s = 2s^2$$

$$9 = s \text{ (since } s > 0)$$

Therefore, the area of $CDEF = s(s − 6) = 9(9 − 6) = 27$ cm$^2$,
and the area of $AHGE = 6(s − 6) = 6(9 − 6) = 18$ cm$^2$.

NOTE: A second solution that uses a quadratic equation is shown on the next page.
Solution 2

We will use the diagram to the right.
Let the side length of $AH$ be $x$.
Since the area of $BFGH = 36 \text{ cm}^2$, then $HB = BF = 6 \text{ cm}$.
Therefore $AE = BF = 6 \text{ cm}$ and $CD = AH + HB = (x + 6) \text{ cm}$.
Since $ABCD$ is a square, $AD = CD = (x + 6) \text{ cm}$ and $ED = AD - AE = (x + 6) - 6 = x \text{ cm}$.

Using the equation:
Area of $BFGH - \text{Area of } CDEF = \text{Area of } CDEF - \text{Area of } AHGE$, we get
\[
36 - x(x + 6) = x(x + 6) - 6x \\
36 - x^2 - 6x = x^2 + 6x - 6x \\
2x^2 + 6x - 36 = 0 \\
2(x^2 + 3x - 18) = 0 \\
2(x + 6)(x - 3) = 0 \\
x = -6 \text{ or } 3 \\
x = 3 \text{ (since } x > 0) 
\]

Therefore the area of $CDEF = x(x + 6) = 3(3 + 6) = 27 \text{ cm}^2$ and area of $AHGE = 6x = 6(3) = 18 \text{ cm}^2$. 
Problem of the Week

Problem D

Three Polygons

In the diagram below, the area of $\triangle ACD$ is twice the area of square $BCDE$. $AC$ and $AD$ meet $BE$ at $K$ and $L$ respectively.

If the side length of the square is 12 cm, determine the area of trapezoid $KCDL$. 

\[ \text{Area of } \triangle ACD = 2 \times \text{Area of square } BCDE \]

\[ \text{Side length of square } = 12 \text{ cm} \]

\[ \text{Area of } KCDL = \text{Area of } \triangle ACD - \text{Area of } \triangle AKL \]

\[ \text{Area of } \triangle AKL = \frac{1}{2} \times \text{Base} \times \text{Height} \]

\[ \text{Base} = \frac{12}{2} = 6 \text{ cm} \]

\[ \text{Height} = \frac{12}{2} = 6 \text{ cm} \]

\[ \text{Area of } \triangle AKL = \frac{1}{2} \times 6 \times 6 = 18 \text{ cm}^2 \]

\[ \text{Area of } KCDL = 2 \times 12^2 - 18 = 288 - 18 = 270 \text{ cm}^2 \]
Problem of the Week
Problem D and Solution
Three Polygons

Problem
In the diagram below, the area of $\triangle ACD$ is twice the area of square $BCDE$. $AC$ and $AD$ meet $BE$ at $K$ and $L$ respectively.

If the side length of the square is 12 cm, determine the area of trapezoid $KCDL$.

Solution
To find the area of a trapezoid, multiply the sum of the lengths of the two parallel sides, $KL$ and $CD$, by the height, $BC$, and divide the product by 2. To solve this problem we need to find the length of $KL$. Let $x$ represent the length of $KL$.

Draw $APQ$ perpendicular to $KL$ and $CD$. It follows that $AP$ is an altitude of $\triangle AKL$ and $AQ$ is an altitude of $\triangle ACD$.

\[
\text{Area of square } BCDE = 12 \times 12 = 144 \text{ cm}^2 \\
\text{Area } \triangle ACD = 2 \times \text{Area of Square } BCDE = 288 \text{ cm}^2 \\
\text{But Area } \triangle ACD = CD \times AQ \div 2 \\
\therefore 288 = 12 \times AQ \div 2 \\
288 = 6(AQ) \\
AQ = 48 \text{ cm}
\]

Since $AQ = 48$ and $PQ = BC = 12$, then $AP = AQ - PQ = 48 - 12 = 36$ cm.

\[
\text{Area of trapezoid } KCDL + \text{Area of } \triangle AKL = \text{Area } \triangle ACD \\
(KL + CD) \times BC \div 2 + KL \times AP \div 2 = 288 \\
(x + 12)(12) \div 2 + x(36) \div 2 = 288 \\
6(x + 12) + 18x = 288 \\
6x + 72 + 18x = 288 \\
24x = 216 \\
x = 9 \text{ cm}
\]

\[
\text{Area of trapezoid } KCDL = \frac{(KL + CD) \times PQ}{2} \\
= \frac{(9 + 12)(12)}{2} \\
= 126 \text{ cm}^2
\]

Therefore the area of trapezoid $KCDL$ is 126 cm$^2$. 
Notes:

1. In order to find the length of $KL$, we could establish that $\triangle ACD \sim \triangle AKL$. From this we can use the fact that the ratio of the altitudes of the two triangles equals the ratio of the corresponding sides in the two similar triangles. The reader may wish to justify this “fact”.

\[
\frac{AP}{AQ} = \frac{KL}{CD} \\
\frac{36}{48} = \frac{x}{12} \\
\frac{3}{4} = \frac{x}{12} \\
\therefore x = 9 \text{ cm}
\]

2. Instead of using the formula to determine the area of the trapezoid, we could find the area by subtracting the area of $\triangle AKL$ from the area of $\triangle ACD$.

\[
\text{Area of trapezoid } KCDL = \text{Area } \triangle ACD - \text{Area of } \triangle AKL \\
= 288 - \frac{(KL)(AP)}{2} \\
= 288 - \frac{9 \times 36}{2} \\
= 288 - 162 \\
= 126 \text{ cm}^2
\]
Problem of the Week
Problem D
Block Walk

A beetle walks on the surface of the $2 \times 3 \times 12$ rectangular prism shown. The beetle wishes to travel from $P$ to $Q$.

What is the length of the shortest path from $P$ to $Q$ that the beetle could take?
Problem of the Week
Problem D and Solution
Block Walk

Problem
A beetle walks on the surface of the $2 \times 3 \times 12$ rectangular prism shown. The beetle wishes to travel from $P$ to $Q$. What is the length of the shortest path from $P$ to $Q$ that the beetle could take?

Solution
Many strategies could be attempted. Perhaps the beetle walks along the edges and travels $12 + 2 + 3 = 15$ units. Perhaps the beetle travels across the right side of the prism from $P$ to the midpoint of the top edge (marked $R$ on the diagram below) and then across the top of the prism to $Q$. Referring to the diagram below, it can be shown that this distance is $PR + RQ = \sqrt{40} + \sqrt{45} \approx 13.03$ units. But is this the shortest distance?

To visualize the possible routes, fold out the sides of the box so that they are laying on the same plane as the top of the box. Label the diagram as shown below. Note that as a result of folding out the sides, corner $P$ appears twice. The second corner is labelled $P'$.

The shortest distance for the beetle to travel is a straight line from $P$ to $Q$ or $P'$ to $Q$. So both cases must be considered.

$PQ$ is the hypotenuse of right-angled triangle $PYQ$. Using Pythagoras’ Theorem,

$$PQ^2 = PY^2 + YQ^2 = 12^2 + 5^2 = 169$$

and $PQ = 13$ follows.

$P'Q$ is the hypotenuse of right-angled triangle $P'XQ$. Using Pythagoras’ Theorem,

$$(P'Q)^2 = (P'X)^2 + XQ^2 = 3^2 + 14^2 = 205$$

and $P'Q = \sqrt{205} \approx 14.31$ follows.

Since $PQ < P'Q$, the shortest distance for the beetle to travel is 13 units on the surface of the block in a straight line from $P$ to $Q$.

This problem is quite straight forward once the three-dimensional nature of the problem is removed.
Problem of the Week
Problem D
More Angle Chasing

The points $A$, $B$, $D$ and $E$ lie on the circumference of a circle with centre $C$. If $\angle BCD = 72^\circ$ and $CD = DE$, determine the measure of $\angle BAE$. 
Problem of the Week
Problem D and Solution
More Angle Chasing

Problem
The points $A$, $B$, $D$ and $E$ lie on the circumference of a circle with centre $C$. If $\angle BCD = 72^\circ$ and $CD = DE$, determine the measure of $\angle BAE$.

Solution
As the solution proceeds, the new markings on the diagram will be explained. Draw radii from $C$ to points $A$ and $E$ on the circumference. Join $B$ to $D$.

Since $CB$ and $CD$ are radii, $CB = CD$ and $\triangle CBD$ is isosceles. Therefore, $\angle CBD = \angle CDB = x^\circ$. But in a triangle the angles sum to $180^\circ$. It follows that $x^\circ + x^\circ + 72^\circ = 180^\circ$. Then $2x^\circ = 108^\circ$ and $x = 54$.

In $\triangle CDE$, $CD = CE$ since they are both radii. But we are given that $CD = DE$. Therefore, $CD = CE = DE$ and $\triangle CDE$ is equilateral. It follows that each angle is $60^\circ$. Therefore, $w = 60$.

Since $CE$ and $CA$ are radii, $CE = CA$ and $\triangle CEA$ is isosceles. Therefore, $\angle CEA = \angle CAE = z^\circ$.

Similarly, since $CA$ and $CB$ are radii, $CA = CB$ and $\triangle CAB$ is isosceles. Therefore, $\angle CAB = \angle CBA = y^\circ$.

The figure $ABDE$ is a quadrilateral and we know that the sum of the interior angles of a quadrilateral is $360^\circ$. Then,

$$\angle BAE + \angle ABD + \angle BDE + \angle DEA = 360^\circ$$

$$(y^\circ + z^\circ) + (y^\circ + x^\circ) + (x^\circ + w^\circ) + (w^\circ + z^\circ) = 360^\circ$$

$$2w^\circ + 2x^\circ + 2y^\circ + 2z^\circ = 360^\circ$$

$$w^\circ + x^\circ + y^\circ + z^\circ = 180^\circ$$

Dividing by 2

$$(60^\circ) + (54^\circ) + y^\circ + z^\circ = 180^\circ$$

Substituting for $w$ and $x$

$$(114^\circ) + y^\circ + z^\circ = 180^\circ$$

$$(y + z)^\circ = 66^\circ$$

$\angle BAE = 66^\circ$ Since $\angle BAE = (y + z)^\circ$

$\therefore \angle BAE = 66^\circ$. 
Problem of the Week
Problem D
A Shade Smaller

The diagonal, $BD$, of rectangle $ABCD$ is divided into 5 equal segments at $W$, $X$, $Y$, and $Z$. If rectangle $ABCD$ has length $AB = 9$ and width $AD = 5$, determine the area of the shaded region.
Problem of the Week
Problem D and Solution
A Shade Smaller

Problem
The diagonal, $BD$, of rectangle $ABCD$ is divided into 5 equal segments at $W$, $X$, $Y$, and $Z$. If rectangle $ABCD$ has length $AB = 9$ and width $AD = 5$, determine the area of the shaded region.

Solution
Solution 1
Using the formula for area of a triangle $= \frac{\text{base} \times \text{height}}{2}$, we have area $\triangle ABD = \frac{9 \times 5}{2} = \frac{45}{2}$ units$^2$.

The triangles $\triangle DAW$, $\triangle WAX$, $\triangle XAY$, $\triangle YAZ$ and $\triangle ZAB$ have the same height. Since $\text{DW} = \text{WX} = \text{XY} = \text{YZ} = \text{ZB}$, the triangles also have equal bases. Therefore, area $\triangle DAW = \text{area} \triangle WAX = \text{area} \triangle XAY = \text{area} \triangle YAZ = \text{area} \triangle ZAB$

$= \frac{1}{5} \left(\text{area} \triangle ABD\right) = \frac{1}{5} \left(\frac{45}{2}\right) = \frac{9}{2}$ units$^2$.

Similarly, the area of $\triangle BCD$ is $\frac{9 \times 5}{2} = \frac{45}{2}$ units$^2$.

The triangles $\triangle DCW$, $\triangle WCX$, $\triangle XCY$, $\triangle YCZ$ and $\triangle ZCB$ have the same height and equal bases. Therefore, area $\triangle DCW = \text{area} \triangle WCX = \text{area} \triangle XCY = \text{area} \triangle YCZ = \text{area} \triangle ZCB$

$= \frac{1}{5} \left(\text{area} \triangle BCD\right) = \frac{1}{5} \left(\frac{45}{2}\right) = \frac{9}{2}$ units$^2$.

Therefore, the area of the shaded region is $4 \left(\frac{9}{2}\right) = 18$ units$^2$.

Solution 2
Since $ABCD$ is a rectangle, the angle at $A$ is $90^\circ$. We can then use the Pythagorean Theorem to calculate $BD^2 = AB^2 + AD^2 = 9^2 + 5^2 = 81 + 25 = 106$, and so $BD = \sqrt{106}$, since $BD > 0$. Therefore, $DW = WX = XY = YZ = ZB = \frac{1}{5}(BD) = \frac{1}{5}\sqrt{106}$.

Using the formula area of a triangle $= \frac{\text{base} \times \text{height}}{2}$, base $AB = 9$ and height $AD = 5$, we can calculate area $\triangle ABD = \frac{9 \times 5}{2} = \frac{45}{2}$ units$^2$.

Let’s treat $BD = \sqrt{106}$ as the base of $\triangle ABD$ and let $h$ be the corresponding height. Since the area of $\triangle ABD$ is $\frac{45}{2}$, then we have $\frac{\sqrt{106} \times h}{2} = \frac{45}{2}$ and so $\sqrt{106} \times h = 45$, thus $h = \frac{45}{\sqrt{106}}$.

$\triangle WAX$ and $\triangle YAZ$ both have height $h = \frac{45}{\sqrt{106}}$ and base $\sqrt{106}$, so
area $\triangle WAX = \text{area} \triangle YAZ = \frac{1}{2} \left(\sqrt{106}\right) \left(\frac{45}{\sqrt{106}}\right) = \frac{9}{2}$ units$^2$.

Similarly, $\triangle WCX$ and $\triangle YCZ$ both have height $h = \frac{45}{\sqrt{106}}$ and base $\sqrt{106}$, so
area $\triangle WCX = \text{area} \triangle YCZ = \frac{1}{2} \left(\sqrt{106}\right) \left(\frac{45}{\sqrt{106}}\right) = \frac{9}{2}$ units$^2$.

Therefore, the area of the shaded region is $4 \left(\frac{9}{2}\right) = 18$ units$^2$. 
Problem of the Week

Problem D

Pi Day Hexagons

Pi Day is an annual celebration of the mathematical constant \( \pi \). Pi Day is observed on March 14 since 3, 1, and 4 are the first three significant digits of \( \pi \).

Archimedes determined lower and upper bounds for \( \pi \) by finding the perimeters of regular polygons inscribed and circumscribed in a circle with a diameter of length 1. (An inscribed polygon of a circle has all vertices on the circle. A circumscribed polygon of a circle has all sides tangent to the circle.) We will determine such bounds by looking at regular hexagons inscribed and circumscribed in a circle with centre \( C \) and diameter 1.

Since the circle has circumference equal to \( \pi \), the perimeter of the inscribed regular hexagon \( DEBGFA \) will give a lower bound for \( \pi \) and the perimeter of the circumscribed regular hexagon \( HIJKLM \) will give an upper bound for \( \pi \).

Using these hexagons, determine a lower and an upper bound for \( \pi \).

Some may find the following facts to be useful:

1. When you drop a perpendicular from a vertex of an equilateral triangle to the opposite side, you bisect the angle and the third side. So if we let one side of the equilateral triangle have length 2, we get the following information (below left).

2. The radius of a circle is perpendicular to a tangent of the circle at the point of tangency (above right).

3. The centres of both the inscribed and circumscribed regular hexagons will be at the centre of the circle, \( C \).
Problem

Archimedes determined lower and upper bounds for $\pi$ by finding the perimeters of regular polygons inscribed and circumscribed in a circle with a diameter of length 1. We will determine such bounds by looking at regular hexagons inscribed and circumscribed in a circle with centre $C$ and diameter 1. Since the circle has circumference equal to $\pi$, the perimeter of the inscribed regular hexagon $DEBGFA$ will give a lower bound of $\pi$ and the perimeter of the circumscribed regular hexagon $HIJKLM$ will give an upper bound of $\pi$. Using these hexagons, determine a lower and an upper bound for $\pi$.

Solution

Solution 1

For the inscribed hexagon, draw line segments $AC$ and $DC$, which are both radii of the circle. Since the diameter of the circle is 1, $AC = DC = \frac{1}{2}$. Since the inscribed hexagon is a regular hexagon with centre $C$, we know that $\triangle ACD$ is equilateral (a justification of this is provided at the end of the solution). Thus, $AD = AC = \frac{1}{2}$, and the perimeter of the inscribed regular hexagon is $6 \times AD = 6 \left(\frac{1}{2}\right) = 3$. This gives us a lower bound for $\pi$.

For the circumscribed hexagon, draw line segments $LC$ and $KC$. Since the circumscribed hexagon is a regular hexagon with centre $C$, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Drop a perpendicular from $C$, meeting $LK$ at $N$. Since $CN$ is a radius of the circle, $CN = \frac{\sqrt{3}}{2}$.

Consider $\triangle PQR$ above, which is an equilateral triangle with side length of 2. Drop a perpendicular from $P$, meeting $QR$ at $S$. Now, $\triangle PSR$ is similar to $\triangle CNK$ since $\angle PRS = \angle CKN = 60^\circ$, $\angle PSR = \angle CNK = 90^\circ$ and $\angle SPR = \angle NCK = \frac{1}{2}(60^\circ) = 30^\circ$.

Therefore, 
\[
\frac{CN}{PS} = \frac{CK}{PR} \quad \frac{\frac{1}{2}}{\sqrt{3}} = \frac{CK}{2} \quad \frac{1}{\sqrt{3}} = CK
\]

Since $\triangle LCK$ is equilateral, $LK = CK = \frac{1}{\sqrt{3}}$. Thus, the perimeter of the regular hexagon is $6 \times LK = 6 \left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} \approx 3.46$. This gives us an upper bound for $\pi$.

Therefore, the value for $\pi$ is between 3 and $\frac{6}{\sqrt{3}} \approx 3.46$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$. 
Solution 2

This solution uses trigonometry.

The calculation of the perimeter of the inscribed hexagon is the same as in Solution 1.

For the circumscribed hexagon, draw line segments $LC$ and $KC$. Since the circumscribed hexagon is a regular hexagon with centre $C$, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Thus, $\angle LKC = 60^\circ$.

Drop a perpendicular from $C$, meeting $LK$ at $N$. In $\triangle CNK$, $\angle NKC = \angle LKC = 60^\circ$. Also, $CN$ is a radius of the circle, so $CN = 0.5$. Since $\angle CNK = 90^\circ$,

$$\sin(\angle NKC) = \frac{CN}{KC}$$

$$\sin(60^\circ) = \frac{0.5}{KC}$$

$$\frac{\sqrt{3}}{2} = \frac{0.5}{KC}$$

$$\sqrt{3}KC = 1$$

$$KC = \frac{1}{\sqrt{3}}$$

But $\triangle LCK$ is equilateral so $LK = KC = \frac{1}{\sqrt{3}}$.

Thus, the perimeter of the circumscribed hexagon is $6 \times LK = 6 \times \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} \approx 3.46$. This gives us an upper bound for $\pi$.

Therefore, the value for $\pi$ is between 3 and $\frac{6}{\sqrt{3}}$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$.

EXTENSION: Archimedes used regular 12-gons, 24-gons, 48-gons and 96-gons to get better approximations for the bounds on $\pi$. Can you?

Equilateral triangle justification:

In the solutions, we used the fact that both $\triangle ACD$ and $\triangle LCK$ are equilateral. In fact, a regular hexagon can be split into six equilateral triangles by drawing line segments from the centre of the hexagon to each vertex, which we will now justify.

Consider a regular hexagon with centre $T$. Draw line segments from $T$ to each vertex. Since $T$ is the centre of the hexagon, $T$ is of equal distance to each vertex of the hexagon. Since the hexagon is a regular hexagon, each side of the hexagon has equal length. Thus, the six resultant triangles are congruent. Therefore, the six central angles are equal and each is equal to $\frac{1}{6}(360^\circ) = 60^\circ$.

Now consider $\triangle STU$. We know that $\angle STU = 60^\circ$. Also, $ST = UT$, so $\triangle STU$ is isosceles and $\angle TSU = \angle TUS = \frac{180^\circ - 60^\circ}{2} = 60^\circ$. Therefore, all three angles in $\triangle STU$ are equal to $60^\circ$ and $\triangle STU$ is equilateral. Since the six triangles in the hexagon are congruent, this tells us that the six triangles are all equilateral.
The vertices of $\triangle ABC$ are each located in the first quadrant. Two of the vertices are $A(2, 1)$ and $B(5, 8)$. The third vertex, $C$, has $x$-coordinate 1.

Two of the vertices, $A$ and $B$, are reflected in the $y$-axis and the third vertex, $C$, is reflected in the $x$-axis. The three image points are collinear. That is, a straight line passes through the three image points.

Determine the coordinates of $C$, the third vertex of the triangle.
Problem of the Week
Problem D and Solution
A Time for Reflection

Problem
The vertices of $\triangle ABC$ are each located in the first quadrant. Two of the vertices are $A(2,1)$ and $B(5,8)$. The third vertex, $C$, has $x$-coordinate 1. Two of the vertices, $A$ and $B$, are reflected in the $y$-axis and the third vertex, $C$, is reflected in the $x$-axis. The three image points are collinear. That is, a straight line passes through the three image points. Determine the coordinates of $C$, the third vertex of the triangle.

Solution
Let the coordinates of $C$ be $(1,c), c > 0$, since the point is in the first quadrant.

When a point is reflected about the $y$-axis, the image point has the same $y$-coordinate and the $x$-coordinate is $-1$ times the pre-image $x$-coordinate. In this case, $A(2,1) \rightarrow A'(−2,1)$ and $B(5,8) \rightarrow B'(−5,8)$.

When a point is reflected about the $x$-axis, the image point has the same $x$-coordinate and the $y$-coordinate is $-1$ times the pre-image $y$-coordinate. In this case, $C(1,c) \rightarrow C'(1,−c)$.

The three image points, $A', B', C'$, are collinear.

Solution 1
In this solution, we will find the equation of the line through the three image points. Two points are enough to uniquely determine a line. We begin by first determining the slope, and then the $y$-intercept.

$$\text{slope}(A'B') = \frac{8 - 1}{-5 + 2} = -\frac{7}{3}$$

Since $A'(-2,1)$ is on the line, we can substitute $x = -2, y = 1$, and $m = -\frac{7}{3}$ into $y = mx + b$.

$$1 = -\frac{7}{3}(-2) + b$$
$$3 = 14 + 3b \quad \text{(Multiply both sides by 3)}$$
$$-11 = 3b$$
$$-\frac{11}{3} = b$$

The equation of the line through the three image points is $y = -\frac{7}{3}x - \frac{11}{3}$. Since the point $C'(1,−c)$ is on the line, we can substitute $x = 1$ and $y = −c$ into the equation to solve for $c$, the $x$-coordinate of point $C$. Then, $−c = -\frac{7}{3}(1) - \frac{11}{3} = -\frac{18}{3} = -6$ and $c = 6$ follows.

Therefore, the coordinates of $C$ are $(1,6)$. 
**Solution 2**

In this solution, we will use the fact that the three image points are collinear.

Since \(A'(-2, 1), B'(-5, 8),\) and \(C'(1, -c)\) are collinear, \(\text{slope}(A'B') = \text{slope}(B'C')\).

\[
\frac{8 - 1}{-5 + 2} = \frac{-c - 8}{1 + 5}
\]

\[
\frac{7}{-3} = \frac{-c - 8}{6}
\]

\[
42 = 3c + 24
\]

\[
18 = 3c
\]

\[
6 = c
\]

Therefore, the coordinates of \(C\) are \((1, 6)\).
The rectangular base of an aquarium is 40 cm by 60 cm, and its height is 30 cm. The aquarium is tilted along $AB$ until the water completely covers the end $ABCD$. At this point, it also covers $\frac{4}{5}$ of the base.

Determine the depth of the water, in centimetres, when the aquarium is level.
Problem of the Week
Problem D and Solution
New Depths

Problem
The rectangular base of an aquarium is 40 cm by 60 cm, and its height is 30 cm. The aquarium is tilted along \( AB \) until the water completely covers the end \( ABCD \). At this point, it also covers \( \frac{4}{5} \) of the base. Determine the depth of the water, in centimetres, when the aquarium is level.

Solution
Let \( E \) be the unnamed corner point on the bottom front of the aquarium such that \( EA = 60 \) cm. Let \( P \) be a point on \( EA \) such that \( AP = \frac{4}{5}(EA) = \frac{4}{5}(60) = 48 \) cm.

When the tank is tilted so that the water completely covers end \( ABCD \), a triangular prism with triangular base \( ADP \) and height 40 cm is created. Note that \( \triangle ADP \) is right angled, so when finding the area of \( \triangle ADP \) we can use \( AP \) as the base and \( AD \) as the height.

\[
\text{Volume of triangular prism} = \text{Area of base } \triangle APD \times \text{height} = \frac{1}{2}(AP)(AD) \times (AB) = \frac{1}{2}(48)(30) \times (40) = 28800 \text{ cm}^3
\]

Let \( h \) represent the height of the water when the tank is sitting level. The volume of the rectangular prism \( h \) cm high by 60 cm wide by 40 cm deep is the same as the volume of the triangular prism formed when the tank is tilted.
So, \( 60 \times 40 \times h = 28800 \) and \( h = 12 \) cm follows.

Therefore, the water is 12 cm deep when the aquarium is sitting level.
Number Sense
&
Algebra

TAKEN ME TO THE COVER
Problem of the Week
Problem D
New Heights

The heights of the ten members of the Riky’s basketball team were measured. The heights, in cm, of his nine teammates are 180, 181, 183, 187, 188, 190, 193, 195, and 196. Riky’s height is also a whole number.

The coach of the team made a mistake when measuring Riky’s height. After this mistake was corrected, both the mean and median of the heights increased by 0.5 cm.

Find all possible correct values for Riky’s height.

The mean of a list of numbers is the sum of the numbers in the list divided by the number of numbers in the list.

The median of a list of numbers is the middle number in the ordered list of numbers when there is an odd number of numbers in the list. The median of a list of numbers is the average of the two middle numbers in the ordered list of numbers when there is an even number of numbers in the list.
Problem of the Week
Problem D and Solution
New Heights

Problem
The heights of the ten members of Riky’s basketball team were measured. The heights, in cm, of his nine teammates are 180, 181, 183, 187, 188, 190, 193, 195, and 196. Riky’s height is also a whole number. The coach of the team made a mistake when measuring Riky’s height. After this mistake was corrected, both the mean and median of the heights increased by 0.5 cm. Find all possible correct values for Riky’s height.

Solution
We will let Riky’s correct height be $R$ and the incorrectly recorded height be $r$. To find a relationship between $R$ and $r$ we will look at the averages of the heights. The average of the heights with the incorrect height, $r$, is
\[
\frac{180+181+183+187+188+190+193+195+196+r}{10} = \frac{1693+r}{10}.
\]
The average with the correct height, $R$, is
\[
\frac{180+181+183+187+188+190+193+195+196+R}{10} = \frac{1693+R}{10}.
\]
Therefore,
\[
\frac{1693 + r}{10} + 0.5 = \frac{1693 + R}{10}
\]
\[
1693 + r + 5 = 1693 + R
\]
\[
r + 5 = R
\]
Let’s look at the median for different values of $r$.
Case 1: If $r \leq 182$, then $R \leq 187$ (since $R = r + 5$).
The median with the incorrect height is $\frac{187+188}{2} = 187.5$. Since $R$ and $r$ are both less than 188, the median with the correct height will remain at 187.5. Therefore, this case is not a possible solution.
Case 2: If $r = 183$, then $R = 188$.
The median with the incorrect height is $\frac{187+188}{2} = 187.5$. Since $R = 188$, the median with the correct height is $\frac{188+188}{2} = 188$, which is an increase of 0.5 in the median. Therefore, this case is one possible solution.
Case 3: If $r = 184$, then $R = 189$.
The median with the incorrect height is $\frac{187+188}{2} = 187.5$. Since $R = 189$, the median with the correct height is $\frac{188+189}{2} = 188.5$, which is an increase of 1 in the median. Therefore, this case is not a possible solution.
Case 4: If $185 \leq r \leq 187$, then $190 \leq R \leq 192$.
The median with the incorrect height is $\frac{187 + 188}{2} = 187.5$. Since $190 \leq R \leq 193$, the median with the correct height is $\frac{188 + 190}{2} = 189$, which is an increase of 1.5 in the median. Therefore, this case is not a possible solution.

Case 5: If $r = 188$, then $R = 193$.
The median with the incorrect height is $\frac{188 + 188}{2} = 188$. Since $R = 193$, the median with the correct height is $\frac{188 + 190}{2} = 189$, which is an increase of 1 in the median. Therefore, this case is not a possible solution.

Case 6: If $r = 189$, then $R = 194$.
The median with the incorrect height is $\frac{188 + 189}{2} = 188.5$. Since $R = 194$, the median with the correct height is $\frac{188 + 190}{2} = 189$, which is an increase of 0.5 in the median. Therefore, this case is one possible solution.

Case 7: If $r \geq 190$, then $R \geq 195$.
The median with the incorrect height is $\frac{188 + 190}{2} = 189$. Since both $r$ and $R$ are greater than or equal to 190 the median with the correct height will remain at $\frac{188 + 190}{2} = 189$. Therefore, this case is not a possible solution.

Therefore, the possible correct heights for Riky are 188 cm or 194 cm.
Problem of the Week
Problem D
Dividing a Square

In the diagram below, the large square $ABCD$ is divided into three regions, a smaller square $BFGH$ and two rectangles $AHGE$ and $CDEF$. The area of square $BFGH$ is $36 \text{ cm}^2$.

We are also given that:

Area of $BFGH − \text{Area of } CDEF = \text{Area of } CDEF − \text{Area of } AHGE$

Determine the areas of rectangle $CDEF$ and rectangle $AHGE$. 
Problem of the Week
Problem D and Solution
Dividing a Square

Problem
The large square $ABCD$ (shown above) is divided into three regions, a smaller square $BFGH$ and two rectangles $AHGE$ and $CDEF$. The area of square $BFGH$ is $36 \text{ cm}^2$.

We are also given that:
Area of $BFGH$ − Area of $CDEF$ = Area of $CDEF$ − Area of $AHGE$

Determine the areas of rectangle $CDEF$ and rectangle $AHGE$.

Solution
Solution 1

We will use the diagram to the right in our solution.
Let the side length of $ABCD$ be $s$ cm.
Since the area of $BFGH = 36 \text{ cm}^2$, then $HB = BF = 6$ cm.
Therefore, $AE = BF = 6$ cm and $AH = AB - HB = (s - 6)$ cm.
Similarly, $ED = (s - 6)$ cm.

Using the given equation,

\[
\text{Area of } BFGH - \text{Area of } CDEF = \text{Area of } CDEF - \text{Area of } AHGE
\]

\[
36 - s(s - 6) = s(s - 6) - 6(s - 6)
\]

\[
36 - s^2 + 6s = s^2 - 6s - 6s + 36
\]

\[
18s = 2s^2
\]

\[
9 = s \text{ (since } s > 0)\]

Therefore, the area of $CDEF = s(s - 6) = 9(9 - 6) = 27 \text{ cm}^2$;
and the area of $AHGE = 6(s - 6) = 6(9 - 6) = 18 \text{ cm}^2$.

NOTE: A second solution that uses a quadratic equation is shown on the next page.
Solution 2

We will use the diagram to the right.

Let the side length of $AH$ be $x$.

Since the area of $BFGH = 36 \text{ cm}^2$, then $HB = BF = 6 \text{ cm}$.
Therefore $AE = BF = 6 \text{ cm}$ and $CD = AH + HB = (x + 6)$ cm.

Since $ABCD$ is a square, $AD = CD = (x + 6)$ cm and $ED = AD - AE = (x + 6) - 6 = x$ cm.

Using the equation:

Area of $BFGH - $ Area of $CDEF = $ Area of $CDEF - $ Area of $AHGE$, we get

\[
36 - x(x + 6) = x(x + 6) - 6x \\
36 - x^2 - 6x = x^2 + 6x - 6x \\
2x^2 + 6x - 36 = 0 \\
2(x^2 + 3x - 18) = 0 \\
2(x + 6)(x - 3) = 0 \\
x = -6 \text{ or } 3 \\
x = 3 \text{ (since } x > 0) 
\]

Therefore the area of $CDEF = x(x + 6) = 3(3 + 6) = 27 \text{ cm}^2$ and area of $AHGE = 6x = 6(3) = 18 \text{ cm}^2$. 
Problem of the Week

Problem D

Different Dice, Same Outcome

When a mathematician says ‘a fair die’, they mean there is an equally likely chance of landing on any face of the die.

A standard six-sided die has its faces marked with the numbers 1, 2, 3, 4, 5, and 6. The die is fair and each number is used exactly once.

A special six-sided die has its faces marked with the numbers 1, 3, 4, 5, 6, and 8. The die is fair and each number is used exactly once.

Is it possible to create a second fair, six-sided die marked so that this die and the first special die can be used together to play a board game like Monopoly? (Numbers on the faces of this new die may appear more than once.)

In other words, does another fair, six-sided die exist so that when this new die and the special die are thrown the sum of the numbers on the top faces of the two dice range from 2 to 12 and the probability of obtaining each sum is the same as it would be if two standard dice had been thrown? If it is possible, what numbers would be on the faces of this new die? If it is not possible, explain why not.
Problem of the Week
Problem D and Solution
Different Dice, Same Outcome

Problem
For mathematicians, a fair die means there is an equally likely chance of landing on any face of the die. A standard six-sided die has its faces marked with the numbers 1, 2, 3, 4, 5, and 6. The die is fair and each number is used exactly once. A special six-sided die has its faces marked with the numbers 1, 3, 4, 5, 6, and 8. The die is fair and each number is used exactly once. Is it possible to create a second fair, six-sided die marked so that this die and the first special die can be used together to play a board game like Monopoly? (Numbers on the faces of this new die may appear more than once.) In other words, does another fair, six-sided die exist so that when this new die and the special die are thrown, the sum of the numbers on the top faces of the two dice range from 2 to 12 and the probability of obtaining each sum is the same as it would be if two standard dice had been thrown? If it is possible, what numbers would be on the faces of this new die? If it is not possible, explain why not.

Solution
We first examine what happens when two standard dice are thrown. For each possible roll, we calculate the sum of numbers on the top faces of the dice, and compile this information in a table.

<table>
<thead>
<tr>
<th>Die 1</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
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<td>2</td>
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<td>4</td>
<td>5</td>
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<td>7</td>
<td>8</td>
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<td>4</td>
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<td>9</td>
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<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
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<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

From this table we can determine the probability of each sum by counting the number of ways to get that sum and dividing by 36, the size of the sample space.

\[
P(\text{total} = 2) = P(\text{total} = 12) = \frac{1}{36} \quad \quad \quad \quad P(\text{total} = 3) = P(\text{total} = 11) = \frac{2}{36}
\]

\[
P(\text{total} = 4) = P(\text{total} = 10) = \frac{3}{36} \quad \quad \quad \quad P(\text{total} = 5) = P(\text{total} = 9) = \frac{4}{36}
\]

\[
P(\text{total} = 6) = P(\text{total} = 8) = \frac{5}{36} \quad \quad \quad \quad P(\text{total} = 7) = \frac{6}{36}
\]

We need our new die and the special die to give these same probabilities when rolled together. Since both the special and the new die have 6 sides, the size of the sample space will still be 6 \times 6 = 36.

We need the sums of the top faces to range from 2 to 12. Since the numbers on our special die are 1, 3, 4, 5, 6 and 8, this means that on our new die, the smallest number must be 1 and the largest number must be 4. Now, let’s examine the particular probabilities.

We need the probability that a sum of 2 is thrown to be \(\frac{1}{36}\). Since the size of the sample space is 36, this means that we need exactly 1 way to get a sum of 2. Therefore, exactly one face on our new die must have the number 1 on it.
We need the probability that a sum of 12 is thrown to be \( \frac{1}{36} \). Again, since the size of the sample space is 36, this means that we need exactly 1 way to get a sum of 12. Therefore, exactly one face on the new die must have the number 4 on it.

Therefore, on the new die the numbers on the six faces range from 1 to 4, and exactly one face has a 1 on it and exactly one face has a 4 on it. Therefore, there must be only 2’s and 3’s on the remaining 4 faces.

We need the probability that a sum of 3 is thrown to be \( \frac{2}{36} \). Since the size of the sample space is 36, this means that there must be exactly 2 ways to get a sum of 3. If our new die has one face with a 2 on it, then there would be one way to get a sum of three: by throwing a 1 on the special die and a 2 on our new die. Therefore, we would need a second 2 on the new die. If there was third face with a 2, then there would be a third way to roll a sum of 3, which is too many.

Similarly, we need the probability that a sum of 11 is thrown to be \( \frac{2}{36} \). Since the size of the sample space is 36, this means that there must be exactly 2 ways to get a sum of 11. If our new die has one face with a 3 on it, then there would be one way to get a sum of eleven: by throwing an 8 on the special die and a 3 on our new die. Therefore, we would need a second 3 on the new die. If there was a third face with a 3, then there would be a third way to roll a sum of 11, which is too many. Therefore, the faces on the new die, if it exists, will be 1, 2, 2, 3, 3, 4.

Let’s check that this new die satisfies the conditions of the problem. For each possible roll with the special die and this new die, we calculate the sum of the numbers on the top faces of the dice and compile this information in a table.

<table>
<thead>
<tr>
<th>Special Die</th>
<th>New Die</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 2 2 3 3 4 4 5</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3 5 5 6 6 7</td>
</tr>
<tr>
<td>3</td>
<td>4 6 6 7 7 8</td>
</tr>
<tr>
<td>4</td>
<td>5 7 7 8 8 9</td>
</tr>
<tr>
<td>5</td>
<td>6 8 8 9 9 10</td>
</tr>
<tr>
<td>6</td>
<td>7 9 10 11 11 12</td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

From this table, we can determine the probability of different sums. For each possible sum, we count the number of ways to get that sum, and divide by 36, the size of the sample space.

\[
\begin{align*}
P(\text{total } = 2) &= P(\text{total } = 12) = \frac{1}{36} \\
P(\text{total } = 3) &= P(\text{total } = 11) = \frac{2}{36} \\
P(\text{total } = 4) &= P(\text{total } = 10) = \frac{3}{36} \\
P(\text{total } = 5) &= P(\text{total } = 9) = \frac{4}{36} \\
P(\text{total } = 6) &= P(\text{total } = 8) = \frac{5}{36} \\
P(\text{total } = 7) &= \frac{6}{36}
\end{align*}
\]

Therefore, a second die exists with faces numbered 1, 2, 2, 3, 3, and 4, such that when this die and the special die are thrown, the probability of obtaining each sum is the same as it would be if two standard dice had been thrown. The existence of this die may be surprising. Can you find another pair of special dice that can do the same thing?
The title, “So Many Dynamos”, is an example of a palindrome, a phrase that is the same when read forwards or backwards. Single words like MOM and BOB are palindromes. Numbers like 7, 414 and 12321 are also palindromes.

Next week, on October 8, 2018, you should greet everyone by saying, “Happy Palindrome Day”. When the date is written in the form d-mm-yyyy, October 8, 2018 is 8102018, a palindrome.

In honour of Palindrome Day, we pose a palindrome problem. Determine the number of six-digit palindromic numbers which are divisible by 15.

It may be helpful to note that a number is divisible by 3 if the sum of its digits is divisible by 3. For example, 15972 is divisible by 3 since $1 + 5 + 9 + 7 + 2 = 24$ and 24 is divisible by 3.
Problem of the Week
Problem D and Solution
So Many Dynamos

Problem
A palindrome is a word, phrase, sentence, or number that reads the same forwards and backwards. Determine the number of six-digit palindromic numbers which are divisible by 15.

Solution
We are looking for a six-digit number of the form $abccba$.

For a number to be divisible by 15, it must be divisible by both 3 and 5.

To be divisible by 5, a number must end in 0 or 5. If the required number ends in 0, it must also begin with 0 in order to be a palindrome. But the number $0bccb0$ is not a six-digit number. Therefore, the number cannot end in 0 and hence must start and end with a 5. The required number looks like $5bccb5$.

For a number to be divisible by 3, the sum of its digits must be divisible by 3. Therefore, we get the sum $10 + 2b + 2c$ must be divisible by 3.

Since $b$ and $c$ are digits, we note that $0 \leq b, c \leq 9$, where $b$ and $c$ are integers. This leads to $10 \leq 10 + 2b + 2c \leq 46$.

The numbers between 10 and 46 that are divisible by 3 are 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, and 45.

If we choose 12 to be the sum of the digits we get the equation $10 + 2b + 2c = 12$. We can simplify this equation to an equivalent equation.

\[
10 + 2b + 2c = 12 \\
2(b + c) = 2 \\
b + c = 1
\]

This equivalent equation has the solutions $b = 1, c = 0$ or $b = 0, c = 1$.

If we choose 15 to be the sum of the digits we get the equation $10 + 2b + 2c = 15$. We can simplify this equation to an equivalent equation.

\[
10 + 2b + 2c = 15 \\
2(b + c) = 5 \\
b + c = 2.5
\]

Since $b$ and $c$ are integers, this equation has no solution. Similarly, for any of the odd sums there will no solutions.
Therefore, we only need to solve the equations with even sums. We will summarize the results in a table.

<table>
<thead>
<tr>
<th>Original Equation</th>
<th>Equivalent Equation</th>
<th>Solutions in the form ((b, c))</th>
<th>Number of Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10 + 2b + 2c = 12)</td>
<td>(b + c = 1)</td>
<td>((0,1),(1,0))</td>
<td>2</td>
</tr>
<tr>
<td>(10 + 2b + 2c = 18)</td>
<td>(b + c = 4)</td>
<td>((0,4),(1,3),(2,2),(3,1),(0,4))</td>
<td>5</td>
</tr>
<tr>
<td>(10 + 2b + 2c = 24)</td>
<td>(b + c = 7)</td>
<td>((0,7),(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(7,0))</td>
<td>8</td>
</tr>
<tr>
<td>(10 + 2b + 2c = 30)</td>
<td>(b + c = 10)</td>
<td>((1,9),(2,8),(3,7),(4,6),(5,5),(6,4),(7,3),(8,2),(9,1))</td>
<td>9</td>
</tr>
<tr>
<td>(10 + 2b + 2c = 36)</td>
<td>(b + c = 13)</td>
<td>((4,9),(5,8),(6,7),(7,6),(8,5),(9,4))</td>
<td>6</td>
</tr>
<tr>
<td>(10 + 2b + 2c = 42)</td>
<td>(b + c = 16)</td>
<td>((7,9),(8,8),(9,7))</td>
<td>3</td>
</tr>
</tbody>
</table>

Therefore, the total number of solutions is \(2 + 5 + 8 + 9 + 6 + 3 = 33\). 
Problem of the Week

Problem D

Water Water NOT Everywhere

The sixteen Islands of Math are represented by the circles shown in the diagram below. The lines between the islands represent bridges. Each bridge connects the two islands attached to it. Two islands are said to be adjacent if they are connected by a bridge.

Not every island has water. The number on each island indicates the number of islands adjacent to it that have water.

For each island, determine whether it has water \((w)\) or no water \((n)\).
Problem of the Week
Problem D and Solution
Water Water NOT Everywhere

Problem
The sixteen Islands of Math are represented by the circles shown in the diagram above. The lines between the islands represent bridges. Each bridge connects the two islands attached to it. Two islands are said to be adjacent if they are connected by a bridge. Not every island has water. The number on each island indicates the number of islands adjacent to it that have water. For each island, determine whether it has water (w) or no water (n).

Solution
We begin with the islands highlighted in Image A. We know that the two islands adjacent to the highlighted island on the left must have water, so we place a w on each. Similarly, we know that the two islands adjacent to the highlighted island on the right must have no water, so we place an n on each. We now look at the island highlighted in Image B. We know one of its adjacent islands has no water. Since the number on the island is 3 and there are only three more adjacent islands, each of these must have water and we place a w on each.
The island highlighted in Image C already has three adjacent islands with water. Therefore, the fourth adjacent island must have no water and we place an $n$ in it. The island highlighted in Image D already has one adjacent island with no water. Since the number on the island is 3 and there are only three more adjacent islands, each of these must have water and we place a $w$ on each.

In Image E, the highlighted island already has four adjacent islands with water. Therefore, the remaining two adjacent islands have no water and we place an $n$ on each. In Image F, the highlighted island needs two adjacent islands with water, currently there is only one. Therefore, the remaining adjacent island must have water and we place a $w$ on it.
The island highlighted in Image G already has two adjacent islands with water. Therefore, the remaining adjacent island has no water and we place an \( n \) on it. In Image H, the highlighted island needs two adjacent islands with water, currently there is only one. Therefore, the remaining adjacent island must have water and we place a \( w \) on it.

![Image G](image_g.png)

![Image H](image_h.png)

In Image J, the highlighted island needs two adjacent islands with water, currently there is only one. Therefore, the remaining adjacent island must have water and we place a \( w \) on it. We have now placed letters on all islands. In Image K, we replace all original numbers and keep all the letters. We check each island and verify the solution. We will leave this as an exercise for the reader.

![Image J](image_j.png)

![Image K](image_k.png)

**Connections to Computer Science**

This task concerns logic and inference. The logic in this problem is to understand that, for example, an island marked as 0 means that none of its neighbours has water. From this fact, we can infer further facts, and build up a solution to the larger problem. The method of building a solution using inference is what is called a bottom-up algorithm: we start with a solution to a very small part of problem (the "bottom" of the problem) and then build up larger and larger solutions until we have solved the entire problem (the "top" of the problem).
Problem of the Week
Problem D
Spare Change

Kees emptied his piggy bank of all its 34 coins with a total value of $5.30. The coins are nickels, dimes or quarters only. There are twice as many quarters as dimes. How many of each type of coin does Kees have?

Note: A nickel is worth 5 cents (5¢), a dime is worth 10 cents (10¢), and a quarter is worth 25 cents (25¢).
Problem of the Week
Problem D and Solution
Spare Change

Problem
Kees emptied his piggy bank of all its 34 coins with a total value of $5.30. The coins are nickels, dimes or quarters only. There are twice as many quarters as dimes. How many of each type of coin does Kees have?

Solution
Let $n$ be the number of nickels, $d$ be the number of dimes and $q$ be the number of quarters.

From the total number of coins we get the equation $n + d + q = 34 \quad (1)$. 

From the value of the coins we get the equation $5n + 10d + 25q = 530 \quad (2)$. 

We also know that $q = 2d \quad (3)$.

Substituting equation (3) into equation (1) and simplifying:

\[
\begin{align*}
  n + d + 2d &= 34 \\
  n + 3d &= 34 \quad (4)
\end{align*}
\]

Substituting equation (3) into equation (2) and simplifying:

\[
\begin{align*}
  5n + 10d + 50d &= 530 \\
  5n + 60d &= 530 \\
  n + 12d &= 106 \quad (5)
\end{align*}
\]

We can isolate $n$ in equation (4) to get $n = 34 - 3d$.

We can isolate $n$ in equation (5) to get $n = 106 - 12d$.

We equate the two $n$'s and solve for $d$:

\[
\begin{align*}
  34 - 3d &= 106 - 12d \\
  -3d + 12d &= 106 - 34 \\
  9d &= 72 \\
  d &= 8
\end{align*}
\]

We now substitute $d = 8$ into equation (4) to solve for $n$:

\[
\begin{align*}
  n + 3d &= 34 \\
  n + 3(8) &= 34 \\
  n + 24 &= 34 \\
  n &= 10
\end{align*}
\]

Finally, substitute $d = 8$ into equation (3) to find $q = 2(8) = 16$.

Therefore, Kees has 10 nickels, 8 dimes and 16 quarters.
Problem of the Week
Problem D
Three Polygons

In the diagram below, the area of $\triangle ACD$ is twice the area of square $BCDE$. $AC$ and $AD$ meet $BE$ at $K$ and $L$ respectively.

If the side length of the square is 12 cm, determine the area of trapezoid $KCDL$. 
Problem of the Week  
Problem D and Solution  
Three Polygons

Problem
In the diagram below, the area of \( \triangle ACD \) is twice the area of square \( BCDE \). \( AC \) and \( AD \) meet \( BE \) at \( K \) and \( L \) respectively.

If the side length of the square is 12 cm, determine the area of trapezoid \( KCDL \).

Solution
To find the area of a trapezoid, multiply the sum of the lengths of the two parallel sides, \( KL \) and \( CD \), by the height, \( BC \), and divide the product by 2. To solve this problem we need to find the length of \( KL \). Let \( x \) represent the length of \( KL \).

Draw \( APQ \) perpendicular to \( KL \) and \( CD \). It follows that \( AP \) is an altitude of \( \triangle AKL \) and \( AQ \) is an altitude of \( \triangle ACD \).

\[
\text{Area of square } BCDE = 12 \times 12 = 144 \text{ cm}^2 \\
\text{Area } \triangle ACD = 2 \times \text{Area of Square } BCDE = 288 \text{ cm}^2 \\
\text{But Area } \triangle ACD = CD \times AQ \div 2 \\
\therefore 288 = 12 \times AQ \div 2 \\
288 = 6(AQ) \\
AQ = 48 \text{ cm}
\]

Since \( AQ = 48 \) and \( PQ = BC = 12 \), then \( AP = AQ - PQ = 48 - 12 = 36 \) cm.

\[
\text{Area of trapezoid } KCDL + \text{Area of } \triangle AKL = \text{Area } \triangle ACD \\
(KL + CD) \times BC \div 2 + KL \times AP \div 2 = 288 \\
(x + 12)(12) \div 2 + x(36) \div 2 = 288 \\
6(x + 12) + 18x = 288 \\
6x + 72 + 18x = 288 \\
24x = 216 \\
x = 9 \text{ cm}
\]

\[
\text{Area of trapezoid } KCDL = \frac{(KL + CD) \times PQ}{2} \\
= \frac{(9 + 12)(12)}{2} \\
= 126 \text{ cm}^2
\]

Therefore the area of trapezoid \( KCDL \) is 126 \( \text{cm}^2 \).
Notes:

1. In order to find the length of $KL$, we could establish that $\triangle ACD \sim \triangle AKL$. From this we can use the fact that the ratio of the altitudes of the two triangles equals the ratio of the corresponding sides in the two similar triangles. The reader may wish to justify this “fact”.

\[
\frac{AP}{AQ} = \frac{KL}{CD}
\]

\[
\frac{36}{48} = \frac{x}{12}
\]

\[
\frac{3}{4} = \frac{x}{12}
\]

\[
\therefore x = 9 \text{ cm}
\]

2. Instead of using the formula to determine the area of the trapezoid, we could find the area by subtracting the area of $\triangle AKL$ from the area of $\triangle ACD$.

Area of trapezoid $KCDL = \text{Area } \triangle ACD - \text{Area of } \triangle AKL$

\[
= 288 - \frac{(KL)(AP)}{2}
\]

\[
= 288 - \frac{9 \times 36}{2}
\]

\[
= 288 - 162
\]

\[
= 126 \text{ cm}^2
\]
Problem of the Week
Problem D
Multiply then Add

The product of two integers is computed. The smaller integer is then added to the product, resulting in the sum 299.

Find the four integer pairs that satisfy the conditions of the problem.
Problem of the Week
Problem D and Solution
Multiply then Add

Problem
The product of two integers is computed. The smaller integer is then added to the product, resulting in the sum 299. Find the four integer pairs that satisfy the conditions of the problem.

Solution
Let $x$ represent the smaller integer. Let $y$ represent the larger integer. We know that $y > x$.

The product of the two integers is $xy$. To this product we add the smaller integer $x$. The resulting expression is $xy + x$. We want to find all possible pairs of integers satisfying the equation $xy + x = 299$.

Factoring the left side of the equation, we obtain $x(y + 1) = 299$. We want the product of two integers to be 299. Either both integers are positive or both integers are negative. The factors of 299 are $\{\pm1, \pm13, \pm23, \pm299\}$.

Then $299 = (-299) \times (-1) = (-23) \times (-13) = 1 \times 299 = 13 \times 23$.

The first integer in each product is the smaller of the two integers and is therefore the value of $x$. The second integer in each pair is larger and corresponds to $y + 1$. So, the value of $y$ is 1 less than the second integer in each of the products.

Therefore, the ordered pairs satisfying $xy + x = 299$ are $(x, y) = (-299, -2)$, $(x, y) = (-23, -14)$, $(x, y) = (1, 298)$, and $(x, y) = (13, 22)$.

We can check the validity of each pair by substituting into the expression $xy + x$ to confirm that the value is 299.

When $x = -299$ and $y = -2$,
$xy + x = (-299)(-2) + (-299) = 598 - 299 = 299$.

When $x = -23$ and $y = -14$, $xy + x = (-23)(-14) + (-23) = 322 - 23 = 299$.

When $x = 1$ and $y = 298$, $xy + x = (1)(298) + (1) = 298 + 1 = 299$.

When $x = 13$ and $y = 22$, $xy + x = (13)(22) + (13) = 286 + 13 = 299$. 


The populations of Dogstown and Catsville were equal at the end of 2015. The population of Dogstown decreased by 3.2% during 2016, then increased by 8.1% during 2017. The population of Catsville increased by 2% during 2016, then increased by $r\%$ during 2017. If the populations of the towns were equal again at the end of 2017, determine the value of $r$ correct to one decimal place.
Problem of the Week
Problem D and Solution
Back to Where We Started

Problem
The populations of Dogstown and Catsville were equal at the end of 2015. The population of Dogstown decreased by 3.2% during 2016, then increased by 8.1% during 2017. The population of Catsville increased by 2% during 2016, then increased by \( r \% \) during 2017. If the populations of the towns were equal again at the end of 2017, determine the value of \( r \) correct to one decimal place.

Solution
Let \( p \) be the population of Dogstown at the end of 2015. Since Dogstown and Catsville have the same population size, then \( p \) is also the population of Catsville at the end of 2015.

The population of Dogstown decreased by 3.2% in 2016, so the population at the end of 2016 is
\[
p - \frac{3.2}{100} p = \left(1 - \frac{3.2}{100}\right) p = 0.968p.
\]
The population of Dogstown then increased by 8.1% during 2017, so the population at the end of 2017 is
\[
0.968p + \left(\frac{8.1}{100}\right) (0.968p) = \left(1 + \frac{8.1}{100}\right)(0.968p) = 1.081(0.968p) = 1.046408p.
\]
The population of Catsville increased by 2.0% in 2016, so the population at the end of 2016 is
\[
p + \frac{2.0}{100} p = \left(1 + \frac{2.0}{100}\right) p = 1.02p.
\]
The population of Catsville then increased by \( r\% \) during 2017, so the population at the end of 2017 is
\[
1.02p + \frac{r}{100}(1.02p) = 1.02p + \frac{1.02rp}{100}.
\]
Since the populations of Dogstown and Catsville are equal at the end of 2017, we have
\[
1.046408p = 1.02p + \frac{1.02rp}{100}
\]
\[
1.046408p - 1.02p = \frac{1.02rp}{100}
\]
\[
0.026408p = \frac{1.02rp}{100}
\]
\[
0.026408 = \frac{1.02r}{100} \implies \text{dividing both sides by } p, \text{ since } p > 0
\]
\[
2.6408 = 1.02r
\]
\[
r \approx 2.6
\]
Therefore, correct to one decimal place, \( r = 2.6 \).
Problem of the Week
Problem D
Cut Along the Dotted Line

Four pieces of lumber are placed in parallel positions, as shown below, perpendicular to the line $W$.

- Piece $A$ is 5 m long and touches $W$
- Piece $B$ is 3 m long and its left end is 3 m from the line $W$
- Piece $C$ is 5 m long and its left end is 2 m from the line $W$
- Piece $D$ is 4 m long and its left end is 1.5 m from the line $W$

A single cut, parallel to $W$, is made along the dotted line $L$. The total length of lumber on each side of $L$ is the same. What is the length, in m, of the part of $A$ to the left of the cut?
Problem of the Week
Problem D and Solution
Cut Along the Dotted Line

Problem

Four pieces of lumber are placed in parallel positions, as shown above, perpendicular to the line \( W \).

- Piece \( A \) is 5 m long and touches \( W \)
- Piece \( B \) is 3 m long and its left end is 3 m from the line \( W \)
- Piece \( C \) is 5 m long and its left end is 2 m from the line \( W \)
- Piece \( D \) is 4 m long and its left end is 1.5 m from the line \( W \)

A single cut, parallel to \( W \), is made along the dotted line \( L \). The total length of lumber on each side of \( L \) is the same. What is the length, in m, of the part of \( A \) to the left of the cut?

Solution

Solution 1

Suppose that the distance from line \( W \) to line \( L \) is \( d \) m. Therefore, the total length of piece \( A \) to the left of the cut is \( d \) m.

Since piece \( B \) is 3 m from line \( W \), then the length of piece \( B \) to the left of \( L \) is \((d - 3)\) m.

Similarly, the lengths of pieces \( C \) and \( D \) to the left of line \( L \) are \((d - 2)\) m and \((d - 1.5)\) m respectively.

Therefore, the total length of lumber to the left of line \( L \) is

\[ d + (d - 3) + (d - 2) + (d - 1.5) = 4d - 6.5 \text{ m}. \]

Since the total length of lumber on each side of the cut is equal, then the length on the left side is also \( \frac{1}{2}(5 + 3 + 5 + 4) = 8.5 \text{ m} \).

Therefore, \( 4d - 6.5 = 8.5 \), or \( 4d = 15 \), or \( d = 3.75 \). Therefore, the length of the part of piece \( A \) to the left of \( L \) is 3.75 m.
Solution 2
Suppose that the distance from line $W$ to line $L$ is $d$ m. Therefore, the total length of piece $A$ to the left of the cut is $d$ m. Since $A$ is 5 m long, the length of $A$ to the right of the cut is $(5 - d)$ m.
Since piece $B$ is 3 m from line $W$, then the length of piece $B$ to the left of $L$ is $(d - 3)$ m. Since $B$ is 3 m long, the length of $B$ to the right of the cut is $3 - (d - 3)$, or $(6 - d)$ m.
Since piece $C$ is 2 m from line $W$, then the length of piece $C$ to the left of $L$ is $(d - 2)$ m. Since $C$ is 5 m long, the length of $C$ to the right of the cut is $5 - (d - 2)$, or $(7 - d)$ m.
Since piece $D$ is 1.5 m from line $W$, then the length of piece $D$ to the left of $L$ is $(d - 1.5)$ m. Since $D$ is 4 m long, the length of $D$ to the right of the cut is $4 - (d - 1.5)$, or $(5.5 - d)$ m.

Therefore, the total length of lumber to the left of line $L$ is
$$d + (d - 3) + (d - 2) + (d - 1.5) = 4d - 6.5$$ m.
The total length of lumber to the right of line $L$ is
$$(5 - d) + (6 - d) + (7 - d) + (5.5 - d) = 23.5 - 4d$$ m.
Since the total length of lumber on each side of the cut is equal, then
$$4d - 6.5 = 23.5 - 4d$$
$$8d = 30$$
$$d = 3.75$$

Therefore, the length of the part of piece $A$ to the left of $L$ is 3.75 m.
Problem of the Week
Problem D
Block Walk

A beetle walks on the surface of the $2 \times 3 \times 12$ rectangular prism shown. The beetle wishes to travel from $P$ to $Q$.

What is the length of the shortest path from $P$ to $Q$ that the beetle could take?
Problem of the Week
Problem D and Solution
Block Walk

Problem
A beetle walks on the surface of the $2 \times 3 \times 12$ rectangular prism shown. The beetle wishes to travel from $P$ to $Q$. What is the length of the shortest path from $P$ to $Q$ that the beetle could take?

Solution
Many strategies could be attempted. Perhaps the beetle walks along the edges and travels $12 + 2 + 3 = 15$ units. Perhaps the beetle travels across the right side of the prism from $P$ to the midpoint of the top edge (marked $R$ on the diagram below) and then across the top of the prism to $Q$. Referring to the diagram below, it can be shown that this distance is $PR + RQ = \sqrt{40} + \sqrt{45} \approx 13.03$ units. But is this the shortest distance?

To visualize the possible routes, fold out the sides of the box so that they are laying on the same plane as the top of the box. Label the diagram as shown below. Note that as a result of folding out the sides, corner $P$ appears twice. The second corner is labelled $P'$.

The shortest distance for the beetle to travel is a straight line from $P$ to $Q$ or $P'$ to $Q$. So both cases must be considered.

$PQ$ is the hypotenuse of right-angled triangle $PYQ$. Using Pythagoras’ Theorem,

$$PQ^2 = PY^2 + YQ^2 = 12^2 + 5^2 = 169 \text{ and } PQ = 13$$

follows.

$P'Q$ is the hypotenuse of right-angled triangle $P'XQ$. Using Pythagoras’ Theorem,

$$(P'Q)^2 = (P'X)^2 + XQ^2 = 3^2 + 14^2 = 205 \text{ and } P'Q = \sqrt{205} \approx 14.31$$

follows.

Since $PQ < P'Q$, the shortest distance for the beetle to travel is 13 units on the surface of the block in a straight line from $P$ to $Q$.

This problem is quite straight forward once the three-dimensional nature of the problem is removed.
In a garden, Levi travels through the labyrinth shown from Entrance to Exit. He is only allowed to travel east, south, southeast, or southwest along a path. (He is never allowed to travel north, northeast, northwest, or west.)

How many different routes can Levi take from Entrance to Exit?
Problem of the Week
Problem D and Solution
More Garden Paths

Problem
In a garden, Levi travels through the labyrinth shown above from Entrance to Exit. He is only allowed to travel east, south, southeast, or southwest along a path. (He is never allowed to travel north, northeast, northwest, or west.) How many different routes can Levi take from Entrance to Exit?

Solution
We begin by labelling the Entrance with the letter $S$ for Start and the Exit with the letter $F$ for Finish. We will then label the other seven intersections in the maze as $A, B, C, D, E, G, H$, as shown. We will also place arrows on the paths to show the direction in which Levi can travel.

We shall then keep track of the number of routes from $S$ to each intersection. We will place the number of routes at each intersection as we go. We will work across three different horizontal levels.

**Level 1**
When we start at $S$ there is only one route to $A$ and only one route to $B$. Therefore, we place a 1 at $A$ to keep track of the number of routes from $S$ to $A$. We also place a 1 at $B$ to keep track of the number of routes from $S$ to $B$. Note: There is 1 route from $S$ to $S$. 
Level 2
To get to $C$, Levi could come directly from $S$ or through $A$. To find the number of routes from $S$ to $C$ we need to sum the number of routes from $S$ to $S$ and the number of routes from $S$ to $A$. Therefore, the number of routes is $1 + 1 = 2$. We place a 2 at $C$ to keep track of the number of routes from $S$ to $C$.

To get to $D$, Levi must come from either $A$, $B$ or $C$. Therefore, the total number of routes from $S$ to $D$ is $1 + 1 + 2 = 4$. We place a 4 at $D$ to keep track of the number of routes from $S$ to $D$.

To get to $E$, Levi must come from $B$ or $D$. Therefore, the total number of routes from $S$ to $E$ is $1 + 4 = 5$. We place a 5 at $E$ to keep track of the number of routes from $S$ to $E$.

In total, there are 17 different routes that Levi can take from Entrance to Exit.

Level 3
To get to $G$, Levi must come from $C$. Therefore, there are only 2 routes from $S$ to $G$. We place a 2 at $G$ to keep track of this.

To get to $H$, Levi must come from either $C$, $D$ or $G$. Therefore, the total number of routes from $S$ to $H$ is $2 + 4 + 2 = 8$. We place an 8 at $H$ to keep track of the number of routes from $S$ to $H$.

To get to $F$, Levi must come from $D$, $E$ or $H$. Therefore, the total number of routes from $S$ to $F$ is $4 + 5 + 8 = 17$. We place a 17 at $F$ to keep track of the number of routes from $S$ to $F$.

In total, there are 17 different routes that Levi can take from Entrance to Exit.
Problem of the Week
Problem D
To Top It Off

Jen, John and Rob ordered two medium pizzas. Each of the medium pizzas is divided into 12 equal sized slices. Each slice can have at most 3 toppings and any two slices of the same pizza may have different toppings. The available toppings are pepperoni, green pepper, mushrooms, onion and pineapple.

Their pizzas were divided as follows:

- Jen ate $\frac{1}{4}$ of all the slices. All of her slices had pepperoni as a topping, two of her slices had mushrooms and two had pineapple.
- John ate $\frac{1}{3}$ of all the slices. All of his slices had mushrooms as a topping, four of his slices also had green pepper and onions.
- Rob ate all of the remaining slices. Four of his slices had only two toppings, onion and pineapple. His remaining slices each had exactly 3 toppings.

If $\frac{1}{2}$ of the slices contained pepperoni, $\frac{1}{4}$ of the slices contained pineapple and $\frac{1}{6}$ of the slices had green pepper, determine the toppings on Rob’s remaining slices.
Problem of the Week
Problem D and Solution
To Top It Off

Problem
Jen, John and Rob ordered two medium pizzas. Each of the medium pizzas is divided into 12 equal sized slices. Each slice can have at most 3 toppings and any two slices of the same pizza may have different toppings. The available toppings are pepperoni, green pepper, mushrooms, onion and pineapple. Their pizzas were divided as follows:

- Jen ate $\frac{1}{4}$ of all the slices. All of her slices had pepperoni as a topping, two of her slices had mushrooms and two had pineapple.
- John ate $\frac{1}{3}$ of all the slices. All of his slices had mushrooms as a topping, four of his slices also had green pepper and onions.
- Rob ate all of the remaining slices. Four of his slices had only two toppings, onion and pineapple. His remaining slices each had exactly 3 toppings.

If $\frac{1}{2}$ of the slices contained pepperoni, $\frac{1}{4}$ of the slices contained pineapple and $\frac{1}{6}$ of the slices had green pepper, determine the toppings on Rob’s remaining slices.

Solution
Jen, John and Rob ordered two pizzas, each with 12 slices. Therefore, they have a total of $2 \times 12 = 24$ slices of pizza.

Jen ate $\frac{1}{4}$ of the total number of slices, so she ate $\frac{1}{4} \times 24 = 6$ slices of pizza. All 6 slices had pepperoni, two also had mushrooms and two also had pineapple.

John ate $\frac{1}{3}$ of the total number of slices, so he ate $\frac{1}{3} \times 24 = 8$ slices of pizza. All 8 slices had mushrooms, four also had green pepper and onions. So four of the slices had mushrooms, green pepper and onions, and 4 had at least mushrooms.

Rob ate all of the remaining slices, so he ate $24 - 6 - 8 = 10$ slices. Four of these slices only had onion and pineapple. We are asked to determine the toppings on Rob’s remaining 6 slices.

We are given that $\frac{1}{2}$ of the total number of slices were covered in pepperoni, so $\frac{1}{2} \times 24 = 12$ slices of pizza had pepperoni. Jen ate 6 of these 12 slices. Therefore, there were 6 other slices with pepperoni on them.

We are also given that $\frac{1}{4}$ of the total number of slices were covered in pineapple, so $\frac{1}{4} \times 24 = 6$ slices of pizza had pineapple. Rob ate 4 of these 6 slices and Jen ate 2. Therefore, there were no other slices with pineapple on them.

Further, we are given that $\frac{1}{6}$ of the total number of slices were covered in green pepper, so $\frac{1}{6} \times 24 = 4$ slices of pizza had green pepper. John ate 4 of these slices. Therefore, there were no other slices with green pepper on them.

Of the five toppings available, pepperoni, green pepper, mushrooms, onion and pineapple, the remaining six slices that Rob ate could not have pineapple or green pepper on them. Since we are also given that these six slices have exactly 3 toppings on them, they must have the toppings pepperoni, mushrooms and onions.

Therefore, Rob’s remaining slices all have the same three toppings: pepperoni, mushrooms and onions.
Problem of the Week
Problem D
It’s a New Year

$5^3$ is a power with base 5 and exponent 3.

$5^3$ means $5 \times 5 \times 5$ and equals 125 when expressed as an integer.

When $8^{672} \times 5^{2019}$ is expressed as an integer, how many digits are in the product?
Problem of the Week
Problem D and Solution
It’s a New Year

Problem

5^3 is a power with base 5 and exponent 3. 5^3 means 5 × 5 × 5 and equals 125 when expressed as an integer. When 8^{672} × 5^{2019} is expressed as an integer, how many digits are in the product?

Solution

An immediate temptation might be to reach for a calculator. In this case, basic calculator technology will let you down. We will look at the problem using our knowledge of powers and corresponding power laws.

\[ 8^{672} × 5^{2019} = (2^3)^{672} × 5^{2019} \]
\[ = 2^{3×672} × 5^{2019} \]
\[ = 2^{2016} × 5^{2019} \]
\[ = 2^{2016} × 5^{2016} × 5^3 \]
\[ = (2 × 5)^{2016} × 125 \]
\[ = 10^{2016} × 125 \]

But 10^{2016} is the number 1 followed by 2016 zeroes. When we multiply this number by the three-digit number 125, we obtain the number 125 followed by 2016 zeroes. Therefore, 8^{672} × 5^{2019} has 2016 + 3 = 2019 digits. Happy New Year again!
Problem of the Week
Problem D
Get Your Tickets

A box contains a total of 400 tickets that come in five colours: blue, green, red, yellow and orange. The ratio of blue to green to red tickets is $1 : 2 : 4$. The ratio of green to yellow to orange tickets is $1 : 3 : 6$. How many tickets are there of each colour?
Problem
A box contains a total of 400 tickets that come in five colours: blue, green, red, yellow and orange. The ratio of blue to green to red tickets is $1 : 2 : 4$. The ratio of green to yellow to orange tickets is $1 : 3 : 6$. How many tickets are there of each colour?

Solution
Solution 1
We denote the number of tickets of each of the five colours by the first letter of the colour. We are given that $b : g : r = 1 : 2 : 4$ and that $g : y : o = 1 : 3 : 6$.

Through multiplication by 2, the ratio $1 : 3 : 6$ is equivalent to the ratio $2 : 6 : 12$.

Thus, $g : y : o = 2 : 6 : 12$.

We chose to scale this ratio by a factor of 2 so that the only colour common to the two given ratios, green, now has the same number in both of these ratios.

That is, $b : g : r = 1 : 2 : 4$ and $g : y : o = 2 : 6 : 12$ and since the term $g$ is 2 in each ratio, then we can combine these to form a single ratio,

$$b : g : r : y : o = 1 : 2 : 4 : 6 : 12$$

This ratio tells us that for every blue ticket, there are 2 green, 4 red, 6 yellow, and 12 orange tickets. Thus, if there was only 1 blue ticket, then there would be $1 + 2 + 4 + 6 + 12 = 25$ tickets in total.

However, we are given that the box contains 400 tickets in total. Therefore, the number of blue tickets in the box is $\frac{400}{25} = 16$.


Therefore, there are 16 blue, 32 green, 64 red, 96 yellow, and 192 orange tickets. (Note that there are $16 + 32 + 64 + 96 + 192 = 400$ tickets in total.)
Solution 2

We denote the number of tickets of each of the five colours by the first letter of the colour. We are given that $b : g : r = 1 : 2 : 4$ and that $g : y : o = 1 : 3 : 6$.

The first ratio tells us that $\frac{b}{g} = \frac{1}{2}$, and so $b = \frac{g}{2}$.

The first ratio also tells us that $\frac{g}{r} = \frac{2}{4}$, and so $r = 2g$.

The second ratio tells us that $\frac{g}{y} = \frac{1}{3}$, and so $y = 3g$.

The second ratio also tells us that $\frac{g}{o} = \frac{1}{6}$, and so $o = 6g$.

We are given that there are a total of 400 tickets. That is, $b + g + r + y + o = 400$. Substituting $b = \frac{g}{2}, r = 2g, y = 3g,$ and $o = 6g$, this becomes

$$\frac{g}{2} + g + 2g + 3g + 6g = 400$$

$$\frac{25}{2}g = 400$$

$$g = 32$$

Thus, $b = \frac{g}{2} = 16$, $r = 2g = 64$, $y = 3g = 96$, and $o = 6g = 192$.

Therefore, there are 16 blue, 32 green, 64 red, 96 yellow, and 192 orange tickets.
The positive integers are written consecutively in groups of seven so that the first row contains the numbers 1, 2, 3, 4, 5, 6, 7; the second row contains the numbers 8, 9, 10, 11, 12, 13, 14; the third row contains the numbers 15, 16, 17, 18, 19, 20, 21; etc.

The row sum of a row is the sum of the numbers in the row. For example, the row sum of the first row is \(1 + 2 + 3 + 4 + 5 + 6 + 7 = 28\).

Determine the numbers in the row that has a row sum closest to 2019.
Problem of the Week
Problem D and Solution
New Year Sum

Problem
The positive integers are written consecutively in groups of seven so that the first row contains the numbers 1, 2, 3, 4, 5, 6, 7; the second row contains the numbers 8, 9, 10, 11, 12, 13, 14; the third row contains the numbers 15, 16, 17, 18, 19, 20, 21; etc. The row sum of a row is the sum of the numbers in the row. For example, the row sum of the first row is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. Determine the numbers in the row that has a row sum closest to 2019.

Solution
Solution 1
Observe that the last number in any row is a multiple of 7. If $n$ is the row number, then the last number in the $n^{th}$ row is $7n$. Since the last number in row $n$ is $7n$, the six preceding numbers in the row are $7n - 1$, $7n - 2$, $7n - 3$, $7n - 4$, $7n - 5$, and $7n - 6$.

The sum of the numbers in the $n^{th}$ row is

$$(7n - 6) + (7n - 5) + (7n - 4) + (7n - 3) + (7n - 2) + (7n - 1) + 7n$$

which simplifies to $49n - 21$. We want to find the integer value of $n$ so that $49n - 21$ is as close to 2019 as possible.

$$49n - 21 = 2019$$
$$49n = 2040$$
$$n \approx 41.6$$

The closest integer to 41.6 is 42. Therefore, $n = 42$ and the row sum is $49n - 21 = 49(42) - 21 = 2037$. The last number in the $42^{nd}$ row is $7 \times 42 = 294$. The seven numbers in the $42^{nd}$ row are 288, 289, 290, 291, 292, 293 and 294. The $41^{st}$ row contains the numbers 281, 282, 283, 284, 285, 286 and 287, and the row sum is 1988. This row sum is farther from 2019 than the $42^{nd}$ row sum of 2037 is.

Therefore, the row with the sum closest to 2019 contains the numbers 288, 289, 290, 291, 292, 293 and 294.

The second solution approaches the problem by establishing a linear relationship.
Solution 2

Let \( x \) represent the row number and \( y \) represent the sum of the numbers in the row. Observe that the seventh number in any row is a multiple of 7. In fact, the seventh number in any row is 7 times the row number or \( 7x \). The following table of values shows the row sums for the first three rows.

<table>
<thead>
<tr>
<th>Row Number</th>
<th>Row Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>77</td>
</tr>
<tr>
<td>3</td>
<td>126</td>
</tr>
</tbody>
</table>

It appears that the \( y \) values increase by 49 as the \( x \) values increase by 1. We will verify that this is true. In solution 1 we saw that the sum of the numbers in the \( n^{\text{th}} \) row is \( 49n - 21 \). Therefore, the sum of the numbers in the \( x^{\text{th}} \) row is \( 49x - 21 \) and the sum of the numbers in the \( (x + 1)^{\text{th}} \) row is \( 49(x + 1) - 21 = (49x - 21) + 49 \). Thus, the \( y \) values increase by 49 as the \( x \) values increase by 1. This tells us that the sum of the fourth row should be \( 126 + 49 = 175 \). We can verify this by adding \( 22 + 23 + 24 + 25 + 26 + 27 + 28 \), the numbers in the fourth row. The sum is indeed 175.

As the values of \( x \) increase by 1, the values of \( y \) increase by 49. The relation is linear. The slope is \( \frac{\Delta y}{\Delta x} = \frac{49}{1} = 49 \). Substituting \( x = 1, \ y = 28, \ m = 49 \) into \( y = mx + b \)

\[
\begin{align*}
28 &= 49(1) + b \\
-21 &= b
\end{align*}
\]

The equation of the line which passes through the points in the relation is \( y = 49x - 21 \). Note that \( x \) and \( y \) are positive integers. We want to find the value of \( x \), the row number, so that the value of \( y \), the row sum, is as close to 2019 as possible.

\[
\begin{align*}
49x - 21 &= 2019 \\
49x &= 2040 \\
x &= 41.6
\end{align*}
\]

We want the integer value for \( x \) that is closest to 41.6. Therefore, \( x = 42 \) and the row sum is \( y = 49(42) - 21 = 2037 \). The row sum when \( x = 41 \) is \( y = 49(41) - 21 = 1988 \). The row sum 2037 is closer to 2019 than the row sum 1988.

The seventh number in the 42\textsuperscript{nd} row is \( 7 \times 42 = 294 \). The seven numbers in the 42\textsuperscript{nd} row are 288, 289, 290, 291, 292, 293, and 294.

Therefore, the row with the sum closest to 2019 contains the numbers 288, 289, 290, 291, 292, 293, and 294, and the row sum is 2037.
Problem of the Week
Problem D
Rotten Fruit

At Math’s Grocers, Alan stocks apples and Pruneet stocks pears. One day they noticed that an equal number of apples and pears were rotten. Also, \( \frac{2}{3} \) of the apples were rotten and \( \frac{3}{4} \) of the pears were rotten. Of the total number of apples and pears, what fraction was rotten?
Problem of the Week
Problem D and Solution
Rotten Fruit

Problem
At Math’s Grocers Alan stocks apples and Pruneet stocks pears. One day they noticed that an equal number of apples and pears were rotten. Also, \( \frac{2}{3} \) of the apples were rotten and \( \frac{3}{4} \) of the pears were rotten. Of the total number of apples and pears, what fraction was rotten?

Solution
Solution 1
Let the total number of apples be represented by \( a \) and the total number of pears be represented be \( p \). Since there were an equal number of rotten apples and rotten pears, then \( \frac{2}{3}a = \frac{3}{4}p \), so \( p = \frac{4}{3}(\frac{2}{3}a) = \frac{8}{9}a \). Therefore, the total number of apples and pears was \( a + p = a + \frac{8}{9}a = \frac{17}{9}a \). Also, the total number of rotten fruit was \( 2(\frac{2}{3}a) = \frac{4}{3}a \). Therefore, the fraction of the total amount of fruit that was rotten was \( \frac{\frac{4}{3}a}{\frac{17}{9}a} = \frac{4}{3} \left( \frac{9}{17} \right) = \frac{12}{17} \).

Solution 2
Since \( \frac{2}{3} \) of the the apples were rotten, \( \frac{3}{4} \) of the pears were rotten, and the number of rotten apples equaled the number of rotten pears, then we could let the number of rotten apples be 6. The number of rotten pears will also be 6. (We choose 6 as it is a multiple of the numerator of each fraction.) If there were 6 rotten apples, then the total number of apples is
\[
6 \div \frac{2}{3} = 6(\frac{3}{2}) = 9.
\]
If there were 6 rotten pears, then the total number of pears is
\[
6 \div \frac{3}{4} = 6(\frac{4}{3}) = 8.
\]
Therefore, there were \( 9 + 8 = 17 \) pieces if fruit in total, of which \( 6 + 6 = 12 \) were rotten. Thus, \( \frac{12}{17} \) of the fruit was rotten.

NOTE: In solution 2 we could have used any multiple 6 for each of the number of rotten apples and the number of rotten pears. The final fraction would eventually reduce to \( \frac{12}{17} \).
Problem of the Week
Problem D
A Shade Smaller

The diagonal, $BD$, of rectangle $ABCD$ is divided into 5 equal segments at $W$, $X$, $Y$, and $Z$. If rectangle $ABCD$ has length $AB = 9$ and width $AD = 5$, determine the area of the shaded region.
Problem of the Week
Problem D and Solution
A Shade Smaller

Problem
The diagonal, $BD$, of rectangle $ABCD$ is divided into 5 equal segments at $W$, $X$, $Y$, and $Z$. If rectangle $ABCD$ has length $AB = 9$ and width $AD = 5$, determine the area of the shaded region.

Solution
Solution 1
Using the formula for area of a triangle $= \frac{\text{base} \times \text{height}}{2}$, we have area $\triangle ABD = \frac{9 \times 5}{2} = \frac{45}{2}$ units$^2$.

The triangles $\triangle DAW$, $\triangle WAX$, $\triangle XAY$, $\triangle YAZ$ and $\triangle ZAB$ have the same height. Since $DW = WX = XY = YZ = ZB$, the triangles also have equal bases. Therefore, area $\triangle DAW = \triangle WAX = \triangle XAY = \triangle YAZ = \triangle ZAB = \frac{1}{5} (\text{area } \triangle ABD) = \frac{1}{5} (\frac{45}{2}) = \frac{9}{2}$ units$^2$.

Similarly, the area of $\triangle BCD$ is $\frac{9 \times 5}{2} = \frac{45}{2}$ units$^2$.

The triangles $\triangle DCW$, $\triangle WCX$, $\triangle XCY$, $\triangle YCZ$ and $\triangle ZCB$ have the same height and equal bases. Therefore, area $\triangle DCW = \triangle WCX = \triangle XCY = \triangle YCZ = \triangle ZCB = \frac{1}{5} (\text{area } \triangle BCD) = \frac{1}{5} (\frac{45}{2}) = \frac{9}{2}$ units$^2$.

Therefore, the area of the shaded region is $4 \left( \frac{9}{2} \right) = 18$ units$^2$.

Solution 2
Since $ABCD$ is a rectangle, the angle at $A$ is $90^\circ$. We can then use the Pythagorean Theorem to calculate $BD^2 = AB^2 + AD^2 = 9^2 + 5^2 = 81 + 25 = 106$, and so $BD = \sqrt{106}$, since $BD > 0$.

Therefore, $DW = WX = XY = YZ = ZB = \frac{1}{5}(BD) = \frac{1}{5}\sqrt{106}$.

Using the formula area of a triangle $= \frac{\text{base} \times \text{height}}{2}$, base $AB = 9$ and height $AD = 5$, we can calculate area $\triangle ABD = \frac{9 \times 5}{2} = \frac{45}{2}$ units$^2$.

Let’s treat $BD = \sqrt{106}$ as the base of $\triangle ABD$ and let $h$ be the corresponding height. Since the area of $\triangle ABD$ is $\frac{45}{2}$, then we have $\frac{\sqrt{106} \times h}{2} = \frac{45}{2}$ and so $\sqrt{106} \times h = 45$, thus $h = \frac{45}{\sqrt{106}}$.

$\triangle WAX$ and $\triangle YAZ$ both have height $h = \frac{45}{\sqrt{106}}$ and base $\frac{\sqrt{106}}{5}$, so area $\triangle WAX = \triangle YAZ = \frac{1}{2} \left( \frac{\sqrt{106}}{5} \right) \left( \frac{45}{\sqrt{106}} \right) = \frac{9}{2}$ units$^2$.

Similarly, $\triangle WCX$ and $\triangle YCZ$ both have height $h = \frac{45}{\sqrt{106}}$ and base $\frac{\sqrt{106}}{5}$, so area $\triangle WCX = \triangle YCZ = \frac{1}{2} \left( \frac{\sqrt{106}}{5} \right) \left( \frac{45}{\sqrt{106}} \right) = \frac{9}{2}$ units$^2$.

Therefore, the area of the shaded region is $4 \left( \frac{9}{2} \right) = 18$ units$^2$. 
Problem of the Week
Problem D
Beep Beep

The game *Beep* is played by a group of people counting up through the positive integers from 1. The first person says “one”, the second “two”, and so on. However, every time a multiple of 9, or a number containing the digit 9 is encountered, to avoid losing, the person must say “beep” instead of stating the number. For example, one part of the game would sound like this: “twelve”, “thirteen”, “fourteen”, “fifteen”, “sixteen”, “seventeen”, “beep”, “beep”, “twenty”. “Eighteen” is replaced by “beep”, since it is a multiple of 9, and “nineteen” is replaced by “beep”, since it contains the digit 9.

What number would they need to make it to in order to have heard “beep” exactly 300 times?

Did you know that a number is divisible by 9 exactly when the sum of its digits is divisible by 9? For example, the number 214578 is divisible by 9 since $2 + 1 + 4 + 5 + 7 + 8 = 27$, which is divisible by 9. In fact, $214578 = 9 \times 23842$. 
Problem of the Week
Problem D and Solution

Beep Beep

Problem
The game *Beep* is played by a group of people counting up through the positive integers from 1. The first person says “one”, the second “two”, and so on. However, every time a multiple of 9, or a number containing the digit 9 is encountered, to avoid losing, the person must say “beep” instead of stating the number. For example, one part of the game would sound like this: “twelve”, “thirteen”, “fourteen”, “fifteen”, “sixteen”, “seventeen”, “beep”, “beep”, “twenty”. What number would they need to make it to in order to have heard “beep” exactly 300 times?

Solution
We first determine the number of integers from 1 to 100 that are replaced by a “beep”.

Since $100 = (11 \times 9) + 1$, there are 11 multiples of 9 between 1 and 100. The integers from 1 to 100 that contain the digit 9 are 9, 19, ..., 79, 89, 90, 91, ..., 97, 98, 99. Thus, there are 19 positive integers from 1 to 100 that contain the digit 9. Some integers that are multiples of 9 will also contain the digit 9, and will have been counted twice. There are 3 integers from 1 to 100 that are multiples of 9 (they are 9, 90, and 99).

Hence, the number of integers from 1 to 100 replaced by a “beep” is $11 + 19 - 3 = 27$.

We now determine the number of integers from 101 to 899 that are replaced by a “beep”.

Since $899 = (99 \times 9) + 8$, there are 99 multiples of 9 between 1 and 899. Since there are 11 multiples of 9 between 1 and 100, there are $99 - 11 = 88$ multiples of 9 between 101 and 899. Also, since there are 19 integers from 1 to 100 that contain the digit 9, there are $19 \times 8 = 152$ integers from 101 to 899 that contain the digit 9.

We now determine the number of integers between 101 and 899 that are both multiples of 9 and contain the digit 9, and thus have been counted twice.

For an integer from 101 to 899 to contain the digit 9, either the second or third digit is 9. Therefore, these numbers are of the form $A9B$ or $AB9$, where $1 \leq A \leq 8$.

For the number to be a multiple of 9, $A + B + 9$ must equal 9, 18 or 27. That is, $A + B$ is equal to 0, 9 or 18.

If $A + B = 0$, then $A = 0$ and $B = 0$, which does not give an integer from 101 to 899.

If $A + B = 18$, then $A = 9$ and $B = 9$, which again does not give an integer from 101 to 899.

Therefore, we need to solve $A + B = 9$. The ordered pairs $(A, B)$ which satisfy $A + B = 9$, $1 \leq A \leq 8$, and $0 \leq B \leq 9$ are $(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1)$.

Each of these ordered pairs gives two different three-digit integers that are multiples of 9. For example, the ordered pair $(1, 8)$ gives the numbers 198 and 189. Therefore, from 101 to 899, there are $2 \times 8 = 16$ integers that are multiples of 9 and contain the digit 9.

Hence, the number of integers replaced by a “beep” from 101 to 899 is $88 + 152 - 16 = 224$.

Thus, from 1 to 899 there will be $27 + 224 = 251$ integers replaced by a “beep”.

Therefore, we need “beep” to be said $300 - 251 = 49$ more times. Beginning at 900, every integer up to 999 contains the digit 9, and thus will be replaced with a “beep”. The 49th integer in this range is 948.

In conclusion, when the number 948 is reached they will have heard “beep” exactly 300 times.
Problem of the Week
Problem D
Pi Day Hexagons

Pi Day is an annual celebration of the mathematical constant $\pi$. Pi Day is observed on March 14 since 3, 1, and 4 are the first three significant digits of $\pi$.

Archimedes determined lower and upper bounds for $\pi$ by finding the perimeters of regular polygons inscribed and circumscribed in a circle with a diameter of length 1. (An inscribed polygon of a circle has all vertices on the circle. A circumscribed polygon of a circle has all sides tangent to the circle.) We will determine such bounds by looking at regular hexagons inscribed and circumscribed in a circle with centre $C$ and diameter 1.

Since the circle has circumference equal to $\pi$, the perimeter of the inscribed regular hexagon $DEBGFA$ will give a lower bound for $\pi$ and the perimeter of the circumscribed regular hexagon $HIJKLM$ will give an upper bound for $\pi$.

Using these hexagons, determine a lower and an upper bound for $\pi$.

Some may find the following facts to be useful:

1. When you drop a perpendicular from a vertex of an equilateral triangle to the opposite side, you bisect the angle and the third side. So if we let one side of the equilateral triangle have length 2, we get the following information (below left).

2. The radius of a circle is perpendicular to a tangent of the circle at the point of tangency (above right).

3. The centres of both the inscribed and circumscribed regular hexagons will be at the centre of the circle, $C$. 

\[
\begin{align*}
Q & \quad S & \quad R \quad \sqrt{3}
\end{align*}
\]
Problem

Archimedes determined lower and upper bounds for $\pi$ by finding the perimeters of regular polygons inscribed and circumscribed in a circle with a diameter of length 1. We will determine such bounds by looking at regular hexagons inscribed and circumscribed in a circle with centre $C$ and diameter 1. Since the circle has circumference equal to $\pi$, the perimeter of the inscribed regular hexagon $DEBGFA$ will give a lower bound of $\pi$ and the perimeter of the circumscribed regular hexagon $HIJKLM$ will give an upper bound of $\pi$. Using these hexagons, determine a lower and an upper bound for $\pi$.

Solution

Solution 1

For the inscribed hexagon, draw line segments $AC$ and $DC$, which are both radii of the circle. Since the diameter of the circle is 1, $AC = DC = \frac{1}{2}$. Since the inscribed hexagon is a regular hexagon with centre $C$, we know that $\triangle ACD$ is equilateral (a justification of this is provided at the end of the solution). Thus, $AD = AC = \frac{1}{2}$, and the perimeter of the inscribed regular hexagon is $6 \times AD = 6 \left(\frac{1}{2}\right) = 3$. This gives us a lower bound for $\pi$.

For the circumscribed hexagon, draw line segments $LC$ and $KC$. Since the circumscribed hexagon is a regular hexagon with centre $C$, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Drop a perpendicular from $C$, meeting $LK$ at $N$. Since $CN$ is a radius of the circle, $CN = \frac{1}{2}$.

Consider $\triangle PQR$ above, which is an equilateral triangle with side length of 2. Drop a perpendicular from $P$, meeting $QR$ at $S$. Now, $\triangle PSR$ is similar to $\triangle CNK$ since $\angle PRS = \angle CKN = 60^\circ$, $\angle PSR = \angle CNK = 90^\circ$ and $\angle SPR = \angle NCK = \frac{1}{2}(60^\circ) = 30^\circ$.

Therefore,

\[
\frac{CN}{PS} = \frac{CK}{PR}
\]

\[
\frac{\frac{1}{2}}{\sqrt{3}} = \frac{CK}{2}
\]

\[
\frac{1}{\sqrt{3}} = CK
\]

Since $\triangle LCK$ is equilateral, $LK = CK = \frac{1}{\sqrt{3}}$. Thus, the perimeter of the regular hexagon is $6 \times LK = 6 \left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} \approx 3.46$. This gives us an upper bound for $\pi$.

Therefore, the value for $\pi$ is between 3 and $\frac{6}{\sqrt{3}} \approx 3.46$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$. 

Solution 2

This solution uses trigonometry.

The calculation of the perimeter of the inscribed hexagon is the same as in Solution 1.

For the circumscribed hexagon, draw line segments $LC$ and $KC$. Since the circumscribed hexagon is a regular hexagon with centre $C$, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Thus, $\angle LKC = 60^\circ$.

Drop a perpendicular from $C$, meeting $LK$ at $N$. In $\triangle CNK$, $\angle NKC = \angle LKC = 60^\circ$. Also, $CN$ is a radius of the circle, so $CN = 0.5$. Since $\angle CNK = 90^\circ$, 

\[
\sin(\angle NKC) = \frac{CN}{KC} \quad \sin(60^\circ) = \frac{0.5}{KC} \quad \frac{\sqrt{3}}{2} = \frac{0.5}{KC} \quad \sqrt{3}KC = 1 \quad KC = \frac{1}{\sqrt{3}}
\]

But $\triangle LCK$ is equilateral so $LK = KC = \frac{1}{\sqrt{3}}$.

Thus, the perimeter of the circumscribed hexagon is $6 \times LK = 6 \times \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} \approx 3.46$. This gives us an upper bound for $\pi$.

Therefore, the value for $\pi$ is between $3$ and $\frac{6}{\sqrt{3}}$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$.

**EXTENSION:** Archimedes used regular 12-gons, 24-gons, 48-gons and 96-gons to get better approximations for the bounds on $\pi$. Can you?

**Equilateral triangle justification:**

In the solutions, we used the fact that both $\triangle ACD$ and $\triangle LCK$ are equilateral. In fact, a regular hexagon can be split into six equilateral triangles by drawing line segments from the centre of the hexagon to each vertex, which we will now justify.

Consider a regular hexagon with centre $T$. Draw line segments from $T$ to each vertex. Since $T$ is the centre of the hexagon, $T$ is of equal distance to each vertex of the hexagon. Since the hexagon is a regular hexagon, each side of the hexagon has equal length. Thus, the six resultant triangles are congruent. Therefore, the six central angles are equal and each is equal to $\frac{1}{6}(360^\circ) = 60^\circ$.

Now consider $\triangle STU$. We know that $\angle STU = 60^\circ$. Also, $ST = UT$, so $\triangle STU$ is isosceles and $\angle TSU = \frac{180^\circ - 60^\circ}{2} = 60^\circ$. Therefore, all three angles in $\triangle STU$ are equal to $60^\circ$ and $\triangle STU$ is equilateral. Since the six triangles in the hexagon are congruent, this tells us that the six triangles are all equilateral.
The number 90 can be expressed as the sum of three consecutive positive integers. That is, \( 90 = 29 + 30 + 31 \). Note that 90 can also be expressed as the sum of one consecutive positive integer, that is \( 90 = 90 \).

How many ways can the number 330 be expressed as the sum of an odd number of consecutive positive integers?
Problem of the Week
Problem D and Solution
Sum Ways

Problem
The number 90 can be expressed as the sum of three consecutive positive integers. That is, \(90 = 29 + 30 + 31\). Note that 90 can also be expressed as the sum of one consecutive positive integer, that is \(90 = 90\). How many ways can the number 330 be expressed as the sum of an odd number of consecutive positive integers?

Solution
We are going to use the following to solve this problem: If there exists an odd number, \(k\), of consecutive integers that sum to 330, then \(k\) is an odd divisor of 330. (We will give a proof of this fact at the end of this solution.)

Furthermore, if \(nk = 330\), then \(n\) is the middle integer in the sum and the first integer is \((n - \frac{k-1}{2})\). We will need to check if this first number is indeed positive.

Since \(330 = 2 \cdot 3 \cdot 5 \cdot 11\), the positive odd divisors of 330 are 1, 3, 5, 11, 15, 33, 55, and 165.

Let’s do the analysis for the divisor 11.

If \(k = 11\), then \(n = \frac{330}{11} = 30\) and the first number in the sum is \(30 - \frac{11-1}{2} = 25\).

We can check that 330 can be expressed as the sum of the positive integers 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, and 35.

In the table below, we analyze each possible odd divisor.

<table>
<thead>
<tr>
<th>Odd Divisor (number of terms in sum)</th>
<th>Middle Number</th>
<th>First Number</th>
<th>Valid Sum?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>330</td>
<td>330</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>110</td>
<td>109</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>66</td>
<td>64</td>
<td>Yes</td>
</tr>
<tr>
<td>11</td>
<td>30</td>
<td>25</td>
<td>Yes</td>
</tr>
<tr>
<td>15</td>
<td>22</td>
<td>15</td>
<td>Yes</td>
</tr>
<tr>
<td>33</td>
<td>10</td>
<td>-6</td>
<td>No</td>
</tr>
<tr>
<td>55</td>
<td>6</td>
<td>-21</td>
<td>No</td>
</tr>
<tr>
<td>165</td>
<td>2</td>
<td>-80</td>
<td>No</td>
</tr>
</tbody>
</table>

Therefore, there are five ways to express 330 as the sum of an odd number of consecutive positive integers. They are:

1) \(330 = 330\)
2) \(330 = 109 + 110 + 111\)
3) \(330 = 64 + 65 + 66 + 67 + 68\)
4) \(330 = 25 + 26 + 27 + 28 + 29 + 30 + 31 + 32 + 33 + 34 + 35\)
5) \(330 = 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29\)
Why does this work?

We will start with the example of $k = 5$.
Five consecutive integers can be expressed as $n - 2, n - 1, n, n + 1,$ and $n + 2$, where $n$ is an integer. Their sum is $(n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 5n$.
Therefore, $5n = 330$ and $n = 66$. Thus, the middle term in the sum is 66, and the sum is $64 + 65 + 66 + 67 + 68 = 330$.

In general, if there are $k$ consecutive integers and $k$ is odd, then we can write the sum of these integers in this way:

$$
\underbrace{(n - k + \frac{1}{2}) + (n - 2) + (n - 1) + n + (n + 1) + (n + 2) + (n + 3) + \cdots + (n + k - \frac{1}{2})}_{\text{k integers}} = kn
$$

Now, we can see that if this sum is equal to 330, then $kn = 330$ and thus $k$ is an odd divisor of 330.

**EXTENSION:** How many ways can the number 330 be expressed as the sum of an EVEN number of consecutive positive integers?
Problem of the Week
Problem D
Adding Digits

$ABC$ is a three-digit integer whose first digit is $A$, second digit is $B$ and third digit is $C$. Similarly, $DEF$ is a three-digit integer whose first digit is $D$, second digit is $E$ and third digit is $F$.

We are given that

\[
\begin{array}{c}
ABC \\
+ DEF \\
\hline
1234
\end{array}
\]

How many different values of $A + B + C + D + E + F$ are there?
Problem of the Week
Problem D and Solution
Adding Digits

Problem

ABC is a three-digit integer whose first digit is A, second digit is B and third digit is C. Similarly, DEF is a three-digit integer whose first digit is D, second digit is E and third digit is F. We are given:

\[
\begin{array}{c}
A \quad B \quad C \\
+ \quad D \quad E \quad F \\
1 \quad 2 \quad 3 \quad 4
\end{array}
\]

How many different values of A + B + C + D + E + F are there?

Solution

To solve this problem, we are going to look at each column starting with the units, then tens, and then finally the hundreds column.

Since C + F ends in a 4, then C + F = 4 or C + F = 14. The value of C + F cannot be 20 or more, as C and F are digits. In the case that C + F = 14, we “carry” a 1 to the tens column.

Since the result in the tens column is 3, then when there is no “carry” from the units column, B + E ends in a 3, and when there is a “carry” from the units column 1 + B + E ends in a 3, so B + E ends in a 2.

If B + E ends in a 3, then B + E = 3 or B + E = 13. The value of B + E cannot be 20 or more, as B and E are digits. In the case that B + E = 13, we “carry” a 1 to the hundreds column.

If B + E ends in a 2, then B + E = 2 or B + E = 12. The value of B + E cannot be 20 or more, as B and E are digits. In the case that B + E = 12, we “carry” a 1 to the hundreds column.

Since the result in the hundreds column is 12, then A + D = 12, or in the case when there was a “carry” from the tens column 1 + A + D = 12, so A + D = 11.

We summarize this information in a tree.

```
Possible Sums

   C + F = 4
   /  \  
  /    \  
B + E = 3  B + E = 13
 /   \        /   
A + D = 12  A + D = 11

   C + F = 14
   /    
  /  \  
B + E = 2  B + E = 12
 /   \        /   
A + D = 12  A + D = 11
```

The branches above give all possible sums.
The first branch has the sum A + B + C + D + E + F = 4 + 3 + 12 = 19.
The second branch has the sum A + B + C + D + E + F = 4 + 13 + 11 = 28.
The third branch has the sum A + B + C + D + E + F = 14 + 2 + 12 = 28.
The fourth branch has the sum A + B + C + D + E + F = 14 + 12 + 11 = 37.

Therefore there are 3 different sums for A + B + C + D + E + F. They are 19, 28, and 37.
Indeed, we can find values for $A, B, C, D, E, F$ that achieve each of these sums. When $A = 9, B = 2, C = 4, D = 3, E = 1, F = 0$, $A + B + C + D + E + F = 19$ and $ABC + DEF = 924 + 310 = 1234$. When $A = 7, B = 9, C = 2, D = 4, E = 4, F = 2$, $A + B + C + D + E + F = 28$ and $ABC + DEF = 792 + 442 = 1234$. When $A = 3, B = 5, C = 8, D = 8, E = 7, F = 6$, $A + B + C + D + E + F = 37$ and $ABC + DEF = 358 + 876 = 1234$. 
Pat rides her bike at 20 km/h and Chris ride his bike at 15 km/h. At noon, Chris is 2 km east of Pat, and each begins to ride east. How many minutes will it take for Pat to catch Chris?
Problem of the Week
Problem D and Solution
Catch Up

Problem
Pat rides her bike at 20 km/h and Chris ride his bike at 15 km/h. At noon, Chris is 2 km east of Pat, and each begins to ride east. How many minutes will it take for Pat to catch Chris?

Solution
For the first two solutions we will use the formula: \( \text{time} = \frac{\text{distance}}{\text{speed}} \)

For the third solution we will use the formula: \( \text{distance} = \text{speed} \times \text{time} \)

Solution 1
Since Pat rides at 20 km/h and Chris rides at 15 km/h, then Pat gains 5 km/h on Chris. Since Chris starts 2 km east of Pat, then it takes Pat \( \frac{2}{5} \) of an hour or \( \frac{2}{5} \times 60 = \frac{120}{5} = 24 \) minutes to catch Chris.

Solution 2
Pat rides at 20 km/h or \( \frac{20}{60} = \frac{1}{3} \) km/min.
Chris rides at 15 km/h or \( \frac{15}{60} = \frac{1}{4} \) km/min.
Therefore Pat gains \( \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \) km/min.
Since Chris starts 2 km east of Pat, then it takes Pat \( 2 \div \frac{1}{12} = 24 \) minutes to catch Chris.

Solution 3
Suppose Pat and Chris meet after \( t \) hours.
Then Pat has biked \( 20 \text{ km/h} \times t \text{ h} = 20t \text{ km} \) and Chris has biked \( 15 \text{ km/h} \times t \text{ h} = 15t \text{ km} \).
Since Chris starts 2 km east of Pat, then when they meet, Pat will have travelled 2 km further than Chris.
That is, \( 20t = 15t + 2 \) or \( 5t = 2 \) or \( t = \frac{2}{5} \) hours = 24 minutes. Therefore, it takes Pat 24 minutes to catch Chris.
Problem of the Week

Problem D

ELEVEN

The word ELEVEN contains four different letters, $E$, $L$, $V$, and $N$. Each letter in the word ELEVEN is assigned a different integer value between 0 and 9, inclusive, to create a six-digit positive integer.

If, for example, $E = 4$, $L = 5$, $V = 6$, and $N = 1$, then the resulting number is 454641. The choice of these digit values for the letters in the word ELEVEN is particularly interesting since the resulting number is divisible by 11.

1. Determine the values of $E$, $L$, $V$, and $N$ which make ELEVEN as large as possible subject to the condition that ELEVEN must be divisible by 11; and

2. determine the values of $E$, $L$, $V$, and $N$ which make ELEVEN as small as possible, subject to the condition that ELEVEN must be divisible by 11.

(Also remember, $E \neq L \neq V \neq N$.)

Did you know that a number is divisible by 11 exactly when the sum of the digits in the odd digit positions minus the sum of the digits in the even digit positions is divisible by 11? The number 454641 is divisible by 11 since 
$$(4 + 4 + 4) - (5 + 6 + 1) = 12 - 12 = 0$$
and 0 is divisible by 11. The number 282623 is also divisible by 11 since 
$$(2 + 2 + 2) - (8 + 6 + 3) = 6 - 17 = -11$$
and $-11$ is divisible by 11.
Problem

The word ELEVEN contains four different letters, $E$, $L$, $V$, and $N$. Each letter in the word ELEVEN is assigned a different integer value between 0 and 9, inclusive, to create a six-digit positive integer. If, for example, $E = 4$, $L = 5$, $V = 6$, and $N = 1$, then the resulting number is 454641. The choice of these digit values for the letters in the word ELEVEN is particularly interesting since the resulting number is divisible by 11.

1. Determine the values of $E$, $L$, $V$, and $N$ which make ELEVEN as large as possible subject to the condition that ELEVEN must be divisible by 11; and
2. determine the values of $E$, $L$, $V$, and $N$ which make ELEVEN as small as possible, subject to the condition that ELEVEN must be divisible by 11. (Also remember, $E \neq L \neq V \neq N$.)

Solution

1. Find the largest six-digit positive integer ELEVEN which is divisible by 11 such that $E \neq L \neq V \neq N$.

In order to make the number as large as possible, we make the leftmost digit as large as possible. In this case, we must make the ten-thousand’s digit $E = 9$. By letting $E = 9$, the resulting number looks like 9$L9V9N$. The sum of the digits in the odd positions would be $9 + 9 + 9 = 27$. We want $27 - (L + V + N)$ to be a multiple of 11.

The next digit to replace is $L$. Again, we will set $L$ equal to the largest possible available number, namely 8. The number now looks like 989$V9N$. We want $27 - (8 + V + N)$ to be a multiple of 11.

Set $V = 7$, the next largest available remaining value. The number looks like 9897$9N$. We want $27 - (8 + 7 + N) = 12 - N$ to be a multiple of 11.

We select the largest value from 0 to 6 so that $12 - N$ is a multiple of 11. The only value that satisfies this is $N = 1$ and the resulting number is 989791. This number equals $11 \times 89981$ and is therefore divisible by 11.
2. Find the smallest six-digit positive integer ELEVEN which is divisible by 11 such that \( E \neq L \neq V \neq N \).

In order to make the number as small as possible, we make the leftmost digit as small as possible. In this case, we must make the ten-thousand’s digit \( E = 1 \). If \( E = 0 \), then we no longer have a six-digit number. The number \( 0L0V0N \) is a five-digit number. By letting \( E = 1 \), the resulting number looks like \( 1L1V1N \). The sum of the digits in the odd positions would be \( 1 + 1 + 1 = 3 \). We want \( 3 - (L + V + N) \) to be a multiple of 11.

The next digit to replace is \( L \). Again we will set \( L \) equal to the smallest possible available number, namely 0. The number now looks like \( 101V1N \). We want \( 3 - (0 + V + N) = 3 - (V + N) \) to be a multiple of 11. If \( V + N = 3 \) then \( 3 - (V + N) = 0 \) and \( 101V1N \) would be a multiple of 11. We could do this if the digits 0 and 3 or 1 and 2 were assigned to \( V \) and \( N \) in some order. But, in either case, one of the digits has already been used. It is not possible to get \( ELEVEN \) to be such that \( E = 1, L = 0, \) and \( V + N = 3 \) so that all of the unknowns have distinct values.

We might be tempted to give up and assign a different value to \( E \). However, we must remember that we are looking for values of \( V \) and \( N \) so that \( 3 - (V + N) \) is a multiple of 11. If \( V + N = 14 \), then

\[
3 - (V + N) = 3 - 14 = -11,
\]

which is a multiple of 11. We want to find a combination of available values from 2 to 9 such that \( V + N = 14 \) and \( V \) is as small as possible. Many combinations exist, namely, 5 and 9, 6 and 8, 7 and 7, 8 and 6, and 9 and 5. We obtain the smallest possible value for \( V \) when \( V = 5 \) and \( N = 9 \). It then follows that the smallest number is \( 101519 \). Since this number is \( 11 \times 9229 = 101519 \), it is divisible by 11.

Therefore, when \( E = 9, L = 8, V = 7, \) and \( N = 1 \), ELEVEN becomes 989791. This is the largest six-digit number satisfying the conditions of the problem.

When \( E = 1, L = 0, V = 5, \) and \( N = 9 \), ELEVEN becomes 101519. This is the smallest six-digit number satisfying the conditions of the problem.
The vertices of $\triangle ABC$ are each located in the first quadrant. Two of the vertices are $A(2, 1)$ and $B(5, 8)$. The third vertex, $C$, has $x$-coordinate 1.

Two of the vertices, $A$ and $B$, are reflected in the $y$-axis and the third vertex, $C$, is reflected in the $x$-axis. The three image points are collinear. That is, a straight line passes through the three image points.

Determine the coordinates of $C$, the third vertex of the triangle.
Problem of the Week
Problem D and Solution
A Time for Reflection

Problem
The vertices of $\triangle ABC$ are each located in the first quadrant. Two of the vertices are $A(2, 1)$ and $B(5, 8)$. The third vertex, $C$, has $x$-coordinate 1. Two of the vertices, $A$ and $B$, are reflected in the $y$-axis and the third vertex, $C$, is reflected in the $x$-axis. The three image points are collinear. That is, a straight line passes through the three image points. Determine the coordinates of $C$, the third vertex of the triangle.

Solution
Let the coordinates of $C$ be $(1, c), c > 0$, since the point is in the first quadrant.

When a point is reflected about the $y$-axis, the image point has the same $y$-coordinate and the $x$-coordinate is $-1$ times the pre-image $x$-coordinate. In this case, $A(2, 1) \rightarrow A'(-2, 1)$ and $B(5, 8) \rightarrow B'(-5, 8)$.

When a point is reflected about the $x$-axis, the image point has the same $x$-coordinate and the $y$-coordinate is $-1$ times the pre-image $y$-coordinate. In this case, $C(1, c) \rightarrow C'(1, -c)$.

The three image points, $A', B', C'$, are collinear.

Solution 1
In this solution, we will find the equation of the line through the three image points. Two points are enough to uniquely determine a line. We begin by first determining the slope, and then the $y$-intercept.

\[
slope(A'B') = \frac{8 - 1}{-5 + 2} = -\frac{7}{3}
\]

Since $A'(-2, 1)$ is on the line, we can substitute $x = -2, y = 1,$ and $m = -\frac{7}{3}$ into $y = mx + b$.

\[
1 = -\frac{7}{3}(-2) + b
\]

\[
3 = 14 + 3b \quad \text{(Multiply both sides by 3)}
\]

\[
-11 = 3b
\]

\[
-\frac{11}{3} = b
\]

The equation of the line through the three image points is $y = -\frac{7}{3}x - \frac{11}{3}$. Since the point $C'(1, -c)$ is on the line, we can substitute $x = 1$ and $y = -c$ into the equation to solve for $c$, the $x$-coordinate of point $C$. Then, $-c = -\frac{7}{3}(1) - \frac{11}{3} = -\frac{18}{3} = -6$ and $c = 6$ follows.

Therefore, the coordinates of $C$ are $(1, 6)$. 
Solution 2

In this solution, we will use the fact that the three image points are collinear.

Since $A'(-2, 1), B'(-5, 8)$, and $C'(1, -c)$ are collinear, $\text{slope}(A'B') = \text{slope}(B'C')$.

\[
\frac{8 - 1}{-5 + 2} = \frac{-c - 8}{1 + 5} \\
\frac{7}{-3} = \frac{-c - 8}{6} \\
42 = 3c + 24 \\
18 = 3c \\
6 = c
\]

Therefore, the coordinates of $C$ are $(1, 6)$. 
Three sequences are illustrated in the following diagram.

The first term in each sequence is 3. In the upper sequence, each term after the first term is 15 more than the previous term. In the middle sequence, each term after the first term is 16 more than the previous term. In the lower sequence, each term after the first term is 18 more than the previous term. The three sequences continue indefinitely.

Of all the terms shown on this page, only the first term is common to all three sequences. What other numbers between 3 and 2019 are common to all three sequences?
Problem
Each of three sequences above starts with the number 3. In the first sequence, each term after the first term is 15 more than the previous term. In the second sequence, each term after the first term is 16 more than the previous term. In the third sequence, each term after the first term is 18 more than the previous term. The three sequences continue indefinitely.
The number 3 is common to all three sequences. What other numbers between 3 and 2019 are common to all three sequences?

Solution
Solution 1
To solve this problem, one could write out each sequence to a term less than or equal to 2019. You would write 135 terms of the first sequence, 127 terms of the second sequence and 113 terms of the third sequence. At this point you would compare the three sequences to find two numbers, 723 and 1443, that are common to all three sequences.

This is not a practical solution if you are solving the problem using pencil and paper. However, a solver could write a computer program that would easily handle this problem.

Solution 2
Notice that the number 147 occurs in both the second and third sequences. This number is $147 - 3 = 144$ greater than the first number common to both sequences. What is the significance of 144? It is the Lowest Common Multiple, or LCM, of 16 and 18. The number 16 written in terms of prime factors is $2 \times 2 \times 2 \times 2$ and the number 18 written in terms of prime factors is $2 \times 3 \times 3$.

A common multiple of both 16 and 18 would be $2 \times 2 \times 2 \times 2 \times 3 \times 3 = 288$.

To find the LCM of 16 and 18 divide 288 by any factors common to both 16 and 18. In this case the only common factor of both 16 and 18 is 2. Therefore, the LCM of 16 and 18 is $288 \div 2 = 144$.

Now, we can create a fourth sequence that starts with 3, and each term after the first term is 144 greater than the previous term. It would look like $3, 147, 291, 435, 579, 723, 867, \ldots$.

What numbers are common to this fourth sequence and the first sequence? We need to find the LCM of 15, the amount that each term in sequence 1 increases by after the first term, and 144, the amount that each term in sequence 4 increases by after the first term. The number 15 written in terms of prime factors is $3 \times 5$ and the number 144 written in terms of prime factors is $2 \times 2 \times 2 \times 3 \times 3$. The number 3 is common to both prime factorizations. So the LCM of 15 and 144 is $15 \times 144 \div 3 = 720$.

Now, we can create a fifth sequence that starts with 3, and each term after the first term is 720 greater than the previous term. It would look like $3, 723, 1443, 2163, \ldots$. This sequence contains all numbers that would be common to each of the three given sequences.

Therefore, there are two numbers between 3 and 2019 common to all three sequences. These numbers are 723 and 1443.
Solution 3

This solution builds on the ideas developed in Solution 2.

To solve the problem, we need to find the Lowest Common Multiple, or LCM, of three numbers, namely 15, 16, and 18. This can be written as LCM(15, 16, 18).

First, write each of the three numbers as a product of their prime factors.

\[
\begin{align*}
15 & = 3 \times 5 \\
16 & = 2 \times 2 \times 2 \times 2 \\
18 & = 2 \times 3 \times 3
\end{align*}
\]

Then, LCM(15, 16, 18) is equal to the product of all the factors of 15, all the factors of 16 that are not common to 15, and all the factors of 18 not listed so far.

So, LCM(15, 16, 18) = (3 × 5) × (2 × 2 × 2 × 2) × (3) = 15 × 16 × 3 = 720.

We will perform the calculation in a different order to verify the calculation.

We could find LCM(16, 18, 15). LCM(16, 18, 15) is equal to the product of all of the factors of 16, all the factors of 18 that are not common to 16, and all the factors of 15 not listed so far.

So, LCM(16, 18, 15) = (2 × 2 × 2 × 2) × (3 × 3) × (5) = 16 × 9 × 5 = 720.

We can now determine the numbers between 3 and 2019 that would be common to all three sequences. The numbers are 3 + 720 = 723 and 723 + 720 = 1443. If we were to continue, the next common number would be 1443 + 720 = 2163 which is greater than 2019.

Therefore, there are two numbers between 3 and 2019 common to all three sequences. These numbers are 723 and 1443.
Problem of the Week
Problem D
Green Thumbs

Two local gardening enthusiasts, Mia Gardner and Ivana Grow, competed for a contract to prepare the gardens at a local park. The contract was awarded to Mia. She worked 9 hours a day for 15 days, and was able to complete \( \frac{3}{8} \) of the entire job.

The Parks Commission wanted to move the job along to complete it in a shorter period of time, so the Commission then hired Ivana Grow to work with Mia. Together they completed the remainder of the job in another 10 days, each working 9 hours per day.

If Ivana had been hired originally instead of Mia, how many hours would it have taken her to complete the entire job on her own?
Problem of the Week
Problem D and Solution
Green Thumbs

Problem
Two local gardening enthusiasts, Mia Gardner and Ivana Grow, competed for a contract to prepare the gardens at a local park. The contract was awarded to Mia. She worked 9 hours a day for 15 days, and was able to complete $\frac{3}{8}$ of the entire job. The Parks Commission wanted to move the job along to complete it in a shorter period of time, so the Commission then hired Ivana Grow to work with Mia. Together they completed the remainder of the job in another 10 days, each working 9 hours per day. If Ivana had been hired originally instead of Mia, how many hours would it have taken her to complete the entire job on her own?

Solution
We must make some reasonable assumptions. Each gardener worked at a constant rate each hour, every day. These rates may or may not have been the same for the two gardeners.

Since Mia completed $\frac{3}{8}$ of the job in 15 days, she would complete $\frac{1}{3}$ of $\frac{3}{8}$ or $\frac{1}{8}$ of the job in 5 days.

Since Mia had completed $\frac{3}{8}$ of the job when Ivana started to work, $\frac{5}{8}$ of the job was left to be completed. Together they completed $\frac{5}{8}$ of the job in 10 days. Since Mia can complete $\frac{1}{8}$ of the job in 5 days, she would have completed $\frac{2}{8}$ of the job in these 10 days. Therefore, Ivana completed the remaining $\frac{5}{8} - \frac{2}{8} = \frac{3}{8}$ of the job in these 10 days.

Since Ivana worked 9 hours a day, this means she completed $\frac{3}{8}$ of the job in $10 \times 9 = 90$ hours. Therefore, she completed $\frac{1}{8}$ of the job in 30 hours. Therefore, she could have completed the entire job on her own in $8 \times 30 = 240$ hours.

For Your Information
Mia completed $\frac{1}{8}$ of the job in 5 days. The whole job could be completed by Mia in $8 \times 5 = 40$ days or 360 hours.

As it was, Mia worked a total of 25 days at 9 hours per day and Ivana worked 10 days at 9 hours per day. They worked a total of $25 \times 9 + 10 \times 9 = 315$ hours.

We know that together, Mia and Ivana completed $\frac{5}{8}$ of the job in 10 days. Then, in 2 days they would have completed $\frac{1}{8}$ of the job and in 16 days they would have completed the entire job. That is, in $16 \times 9 = 144$ hours, working together from the start they would have completed the job.
The rectangular base of an aquarium is 40 cm by 60 cm, and its height is 30 cm. The aquarium is tilted along $AB$ until the water completely covers the end $ABCD$. At this point, it also covers $\frac{4}{5}$ of the base.

Determine the depth of the water, in centimetres, when the aquarium is level.
Problem of the Week
Problem D and Solution
New Depths

Problem
The rectangular base of an aquarium is 40 cm by 60 cm, and its height is 30 cm. The aquarium is tilted along $AB$ until the water completely covers the end $ABCD$. At this point, it also covers $\frac{4}{5}$ of the base. Determine the depth of the water, in centimetres, when the aquarium is level.

Solution
Let $E$ be the unnamed corner point on the bottom front of the aquarium such that $EA = 60$ cm. Let $P$ be a point on $EA$ such that $AP = \frac{4}{5}(EA) = \frac{4}{5}(60) = 48$ cm.

When the tank is tilted so that the water completely covers end $ABCD$, a triangular prism with triangular base $ADP$ and height 40 cm is created. Note that $\triangle ADP$ is right angled, so when finding the area of $\triangle ADP$ we can use $AP$ as the base and $AD$ as the height.

$$\text{Volume of triangular prism} = \text{Area of base } \triangle APD \times \text{height}$$
$$= \frac{1}{2}(AP)(AD) \times (AB)$$
$$= \frac{1}{2}(48)(30) \times (40)$$
$$= 28800 \text{ cm}^3$$

Let $h$ represent the height of the water when the tank is sitting level. The volume of the rectangular prism $h$ cm high by 60 cm wide by 40 cm deep is the same as the volume of the triangular prism formed when the tank is tilted. So, $60 \times 40 \times h = 28800$ and $h = 12$ cm follows.

Therefore, the water is 12 cm deep when the aquarium is sitting level.
Problem of the Week
Problem D
New Year Sum

The positive integers are written consecutively in groups of seven so that the first row contains the numbers 1, 2, 3, 4, 5, 6, 7; the second row contains the numbers 8, 9, 10, 11, 12, 13, 14; the third row contains the numbers 15, 16, 17, 18, 19, 20, 21; etc.

The row sum of a row is the sum of the numbers in the row. For example, the row sum of the first row is \(1 + 2 + 3 + 4 + 5 + 6 + 7 = 28\).

Determine the numbers in the row that has a row sum closest to 2019.

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 & 12 & 13 & 14 \\
15 & 16 & 17 & 18 & 19 & 20 & 21 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
Problem of the Week
Problem D and Solution
New Year Sum

Problem
The positive integers are written consecutively in groups of seven so that the first row contains the numbers 1, 2, 3, 4, 5, 6, 7; the second row contains the numbers 8, 9, 10, 11, 12, 13, 14; the third row contains the numbers 15, 16, 17, 18, 19, 20, 21; etc. The row sum of a row is the sum of the numbers in the row. For example, the row sum of the first row is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. Determine the numbers in the row that has a row sum closest to 2019.

Solution
Solution 1
Observe that the last number in any row is a multiple of 7. If $n$ is the row number, then the last number in the $n^{th}$ row is $7n$. Since the last number in row $n$ is $7n$, the six preceding numbers in the row are $7n - 1$, $7n - 2$, $7n - 3$, $7n - 4$, $7n - 5$, and $7n - 6$.

The sum of the numbers in the $n^{th}$ row is

$$(7n - 6) + (7n - 5) + (7n - 4) + (7n - 3) + (7n - 2) + (7n - 1) + 7n$$

which simplifies to $49n - 21$. We want to find the integer value of $n$ so that $49n - 21$ is as close to 2019 as possible.

$$49n - 21 = 2019$$

$$49n = 2040$$

$$n \approx 41.6$$

The closest integer to 41.6 is 42. Therefore, $n = 42$ and the row sum is $49n - 21 = 49(42) - 21 = 2037$. The last number in the 42nd row is $7 \times 42 = 294$. The seven numbers in the 42nd row are 288, 289, 290, 291, 292, 293 and 294. The 41st row contains the numbers 281, 282, 283, 284, 285, 286 and 287, and the row sum is 1988. This row sum is farther from 2019 than the 42nd row sum of 2037 is.

Therefore, the row with the sum closest to 2019 contains the numbers 288, 289, 290, 291, 292, 293 and 294.

The second solution approaches the problem by establishing a linear relationship.
Let $x$ represent the row number and $y$ represent the sum of the numbers in the row. Observe that the seventh number in any row is a multiple of 7. In fact, the seventh number in any row is 7 times the row number or $7x$. The following table of values shows the row sums for the first three rows.

<table>
<thead>
<tr>
<th>Row Number</th>
<th>Row Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>1</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>77</td>
</tr>
<tr>
<td>3</td>
<td>126</td>
</tr>
</tbody>
</table>

It appears that the $y$ values increase by 49 as the $x$ values increase by 1. We will verify that this is true. In solution 1 we saw that the sum of the numbers in the $n^{th}$ row is $49n - 21$. Therefore, the sum of the numbers in the $x^{th}$ row is $49x - 21$ and the sum of the numbers in the $(x + 1)^{th}$ row is $49(x + 1) - 21 = (49x - 21) + 49$. Thus, the $y$ values increase by 49 as the $x$ values increase by 1. This tells us that the sum of the fourth row should be $126 + 49 = 175$. We can verify this by adding $22 + 23 + 24 + 25 + 26 + 27 + 28$, the numbers in the fourth row. The sum is indeed 175.

As the values of $x$ increase by 1, the values of $y$ increase by 49. The relation is linear. The slope is $\Delta y \over \Delta x = 49 \over 1 = 49$. Substituting $x = 1$, $y = 28$, $m = 49$ into

$$y = mx + b$$

$$28 = 49(1) + b$$

$$-21 = b$$

The equation of the line which passes through the points in the relation is $y = 49x - 21$. Note that $x$ and $y$ are positive integers. We want to find the value of $x$, the row number, so that the value of $y$, the row sum, is as close to 2019 as possible.

$$49x - 21 = 2019$$

$$49x = 2040$$

$$x = 41.6$$

We want the integer value for $x$ that is closest to 41.6. Therefore, $x = 42$ and the row sum is $y = 49(42) - 21 = 2037$. The row sum when $x = 41$ is $y = 49(41) - 21 = 1988$. The row sum 2037 is closer to 2019 than the row sum 1988.

The seventh number in the $42^{nd}$ row is $7 \times 42 = 294$. The seven numbers in the $42^{nd}$ row are 288, 289, 290, 291, 292, 293, and 294.

Therefore, the row with the sum closest to 2019 contains the numbers 288, 289, 290, 291, 292, 293, and 294, and the row sum is 2037.