The problems in this booklet are organized into strands. A problem often appears in multiple strands. The problems are suitable for most students in Grade 11 or higher.
Algebra
&
Equations
Problem of the Week
Problem E
An Uphill Struggle

The following information is known about $\triangle OBC$:

- $O$ is at the origin, and points $B$ and $C$ lie in the first quadrant;
- $\triangle OBC$ is an isosceles right triangle with $OB = BC$ and $\angle OBC = 90^\circ$; and
- the hypotenuse $OC$ is on a line segment with slope 3.

Determine the slope of line segment $OB$. 

![Diagram of triangle OBC with points O, B, and C labeled.]
Problem of the Week
Problem E and Solution
An Uphill Struggle

Problem
The following information is known about $\triangle OBC$: $O$ is at the origin and points $B$ and $C$ lie in the first quadrant; $\triangle OBC$ is an isosceles right triangle with $OB = BC$ and $\angle OBC = 90^\circ$; and the hypotenuse $OC$ is on a line segment with slope 3. Determine the slope of line segment $OB$.

Solution
We present three solutions. The first involves a construction. The second solution follows after making an assumption. The third solution uses trigonometry. The formula used in the third solution may not be familiar to all students.

Solution 1
Draw a line through $C$ parallel to the $x$-axis, intersecting the $y$-axis at $R$.
Draw a line through $B$ parallel to the $y$-axis, intersecting the $x$-axis at $P$ and intersecting the first line through $R$ and $C$ at $Q$.
This construction creates rectangle $OPQR$.
In $\triangle CQB$, let $\angle QCB = \alpha$ and $\angle QBC = \beta$. Since $OPQR$ is a rectangle, $\angle BQC = 90^\circ$ and $\triangle CQB$ is a right angled triangle. It follows that $\alpha + \beta = 90^\circ$.
$\angle QBP$ is a straight angle so $\angle QBC + \angle CBO + \angle OBP = 180^\circ$.
Substituting, we obtain $\beta + 90^\circ + \angle OBP = 180^\circ$ which simplifies to $\beta + \angle OBP = 90^\circ$.
But $\alpha + \beta = 90^\circ$ so it follows that $\angle OBP = \alpha$. Then in right triangle $BPO$, we get $\angle BOP = \beta$.
In $\triangle CQB$ and $\triangle BPO$, since $\angle QCB = \angle OBP = \alpha$, $\angle QBC = \angle BOP = \beta$, and $BC = OB$ (given), then $\triangle CQB \cong \triangle BPO$.
From the triangle congruence, we get $CQ = BP = b$ and $QB = OP = a$.
In rectangle $OPQR$, $RC + CQ = OP$. Substituting, we obtain $RC + b = a$ and $RC = a - b$ follows.
All of this information is shown on the diagram above.
The coordinates of $C$ are $(a - b, a + b)$ and the coordinates of $B$ are $(a, b)$.
We know the slope of $OC = 3$, so $\frac{a + b}{a - b} = 3$. Simplifying, we obtain $a + b = 3a - 3b$ and $a = 2b$ follows.
Then the slope of $OB = \frac{b}{a} = \frac{b}{2b} = \frac{1}{2}$. 
Solution 2

Since $OC$ is a line segment with slope 3, with $O$ at the origin and $C$ in the first quadrant, the coordinates of $C$ will be of the form $(a, 3a)$, where $a$ is some positive number. We will do our calculations with $a = 2$. Then the length of $OC$ is $2\sqrt{10}$. Let $B$ be the point $(p, q)$.

Let $M$ be the midpoint of $OC$. Then $M$ is the point $(1, 3)$. It follows that $OM = MC = \frac{1}{2}OC = \sqrt{10}$.

In an isosceles right triangle, the line segment joining the midpoint of the hypotenuse to the opposite vertex is perpendicular to the hypotenuse and has length equal to half the length of the hypotenuse. (If this result is not known, it is easily shown using congruent triangles.)

It follows that $MB \perp OC$ and $MB = \sqrt{10}$.

Since $MB \perp OC$ and the slope of $OC$ is 3, then the slope of $MB$ is $-\frac{1}{3}$. We can find the equation of the line containing $M(1, 3)$ with slope $-\frac{1}{3}$ by substituting into $y = mx + b$.

$$
\begin{align*}
3 &= -\frac{1}{3}(1) + b \\
9 &= -1 + 3b \\
10 &= 3b \\
\frac{10}{3} &= b
\end{align*}
$$

The equation of the line containing $MB$ is $y = -\frac{1}{3}x + \frac{10}{3}$.

Since $B(p, q)$ is on this line, $q = -\frac{1}{3}p + \frac{10}{3}$. (1)

The length of $MB$ is $\sqrt{10}$. Using $M(1, 3)$ and $B(p, -\frac{1}{3}p + \frac{10}{3})$,

$$
\begin{align*}
MB^2 &= (p - 1)^2 + \left(-\frac{1}{3}p + \frac{10}{3} - 3\right)^2 \\
(\sqrt{10})^2 &= (p - 1)^2 + \left(-\frac{1}{3}p + \frac{1}{3}\right)^2 \\
10 &= (p - 1)^2 + \left(-\frac{1}{3}(p - 1)\right)^2 \\
10 &= (p - 1)^2 + \frac{1}{9}(p - 1)^2 \\
10 &= \frac{10}{9}(p - 1)^2 \\
9 &= (p - 1)^2 \\
\pm 3 &= p - 1
\end{align*}
$$

It follows that $p = 4$ or $p = -2$. Since $B$ is in quadrant 1, $p = -2$ is inadmissible. Therefore, $p = 4$. Substituting in (1), $q = 2$ and $B$ is the point $(4, 2)$. Thus, the slope of $OB = \frac{2}{4} = \frac{1}{2}$.
Solution 3

Since $\triangle OBC$ is an isosceles right triangle with $\angle OBC = 90^\circ$, then $\angle BOC = \angle BCO = 45^\circ$.

Let $\theta$ represent the angle that $OC$ makes with the positive $x$-axis. Since the slope of $OC = 3$, then $\tan \theta = 3$, since the tangent of an angle is equal to the slope of a line that makes this angle with the horizontal (the positive $x$-axis in this case).

The angle that $OB$ makes with the positive $x$-axis is $\theta - 45^\circ$. The slope of $OB$ will equal $\tan(\theta - 45^\circ)$.

Using the fact that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$,

$$
\tan(\theta - 45^\circ) = \frac{\tan \theta - \tan 45^\circ}{1 + \tan \theta \tan 45^\circ} = \frac{3 - 1}{1 + 3(1)} = \frac{2}{4} = \frac{1}{2}
$$

Therefore, the slope of $OB = \tan(\theta - 45^\circ) = \frac{1}{2}$. 
A maple tree is surrounded by two pine trees and two palm trees, as follows:

Five types of bananas, $P, Q, R, S, T$ are placed in the trees. Each tree has exactly one type of banana in it and there is a different type of banana in each tree. A monkey is hungry and takes the same amount of time to eat any banana. The monkey starts on a tree and begins by eating a banana on that tree. The monkey then swings to another tree and eats a banana on that tree. The monkey then swings to another tree and eats a banana on that tree. This will continue until the monkey stops.

It takes the monkey

- three seconds to swing from the maple tree to any other tree or vice versa,
- two seconds to swing from a pine tree to a palm tree or vice versa, and
- seven seconds to swing between two pine trees or two palm trees while avoiding the maple tree along the way.

The monkey eats bananas of type $P, Q, S, R, T, R, P$ in that order then stops. List all the possible types of banana that can be in the maple tree if the total amount of time the monkey swings is as small as possible.
Problem

A maple tree is surrounded by two pine trees and two palm trees, as shown above. Five types of bananas, $P, Q, R, S, T$ are placed in the trees. Each tree has exactly one type of banana in it and there is a different type of banana in each tree. A monkey is hungry and takes the same amount of time to eat any banana. The monkey starts on a tree and begins by eating a banana on that tree. The monkey then swings to another tree and eats a banana on that tree. The monkey then swings to another tree and eats a banana on that tree. This will continue until the monkey stops. It takes the monkey three seconds to swing from the maple tree to any other tree or vice versa, two seconds to swing from a pine tree to a palm tree or vice versa, and seven seconds to swing between two pine trees or two palm trees while avoiding the maple tree along the way. The monkey eats bananas of type $P, Q, S, R, T, R, P$ in that order then stops. List all the possible types of banana can be in the maple tree if the total amount of time the monkey swings is as small as possible.

Solution

The given sequence $P, Q, S, R, T, R, P$ visits all the banana types with six swings $P - Q, Q - S, S - R, R - T, T - R, R - P$. For this sequence, the monkey swings to each tree at least once and from each tree at least once. Therefore, there needs to be a swing to and a swing from the maple. So the minimum time that is possible occurs when there are only two swings to and from the maple and the other four swings are between the pine and palm trees. Thus, the minimum possible time is $2(3) + 4(2) = 14$ seconds. We can show three such routes below.

On the next page we will show why it is not possible to have a 14 second route if the bananas on the maple tree are of type $R$ or $Q$. 
If the bananas on the maple are of type $R$ then there are four swings to or from the maple and the minimum time is now $4(3) + 2(2) = 16$. This is more than 14 seconds. Therefore, if the bananas on the maple are of type $R$, we cannot achieve the minimum time of 14 seconds.

If the bananas on the maple tree are of type $Q$, then to obtain a total time of 14 seconds, the monkey must take two seconds to swing from each of $S$ to $R$, $R$ to $T$ and $R$ to $P$.

This means that the bananas $S$, $T$, and $P$ are all palm trees or all pine trees, which contradicts the initial situation.

Another way of thinking about this is that since the monkey will swing from each of $S$ to $R$, $R$ to $T$ and $R$ to $P$ then at least one of the swings must take 7 s and the minimum time is $2(3) + 3(2) + 7 = 19$ seconds. This is more than 14 seconds. Therefore, if the bananas on the maple are of type $Q$, we cannot achieve the minimum time of 14 seconds.

In conclusion, the bananas on the maple tree could be type $P$, $S$, or $T$.

**Applications to Computer Science**

This problem involves finding the best, or optimal, solution to a problem. Computers are often used to find the maximum or minimum value of some measurement. In this case, we might think of the trees as applications on a touch screen and the monkey swinging as the movement of a human finger from one application to another. A user interface designer might be interested in how to arrange the applications for a common sequence of operations so as to require as little time as possible.
Problem of the Week
Problem E
Many Possibilities?

Two distinct positive integers are multiplied together. This product is then added to the sum of the original two integers resulting in a sum of 195.

Determine all possible pairs of integers which satisfy the above conditions.
Problem of the Week
Problem E and Solution
Many Possibilities?

Problem
Two distinct positive integers are multiplied together. This product is then added to the sum of the original two integers resulting in a sum of 195.

Determine all possible pairs of integers which satisfy all of the conditions.

Solution
Let \( x \) and \( y \) represent the two positive integers. Since the integers are distinct, \( x \neq y \).

The product of the two integers is \( xy \) and the sum is \( (x + y) \).

We want to find all pairs of integers, \( x, y \), such that \( xy + x + y = 195 \).

If we attempt to factor the left side of the equation, we discover that there is no common factor between all three terms. However, if we factor a common factor out of the first two terms on the left side of the equation, we obtain \( x(y + 1) + y = 195 \).

If we add 1 to both sides, we obtain:
\[
(x + 1)(y + 1) = 196.
\]

Now, the left side has a common factor of \( (y + 1) \). After factoring, we obtain:

Looking strictly at the equation \( (x + 1)(y + 1) = 196 \), we see that we are looking for a pair of integers whose product is 196. Using the factors of 196, we obtain
\[
196 = 1 \times 196 = 2 \times 98 = 4 \times 49 = 7 \times 28 = 14 \times 14.
\]

We could also list the first four products in reverse order but the pairs of numbers producing these products will be the same as the pairs producing the first four products already listed.

The numbers in the products are each one more than the numbers we are looking for.

For the product 196 = 1 \times 196, \( x = 0 \) and \( y = 195 \). Since the required numbers are positive integers, this solution is inadmissible.

For the product 196 = 2 \times 98, \( x = 1 \) and \( y = 97 \). This is a valid solution.

For the product 196 = 4 \times 49, \( x = 3 \) and \( y = 48 \). This is a valid solution.

For the product 196 = 7 \times 28, \( x = 6 \) and \( y = 27 \). This is a valid solution.

For the product 196 = 14 \times 14, \( x = 13 \) and \( y = 13 \). Since the required numbers are distinct positive integers, this solution is inadmissible.

There are three pairs of distinct positive integers for which their product and their sum add to 195. The pairs are 1 and 97, 3 and 48, and 6 and 27. A verification is provided on the next page.
<table>
<thead>
<tr>
<th>First Positive Integer</th>
<th>Second Positive Integer</th>
<th>Product</th>
<th>Sum</th>
<th>Product + Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>97</td>
<td>97</td>
<td>98</td>
<td>97 + 98 = 195</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>144</td>
<td>51</td>
<td>144 + 51 = 195</td>
</tr>
<tr>
<td>6</td>
<td>27</td>
<td>162</td>
<td>33</td>
<td>162 + 33 = 195</td>
</tr>
</tbody>
</table>

Note: if the two positive integers did not need to be distinct, then $x = 13$ and $y = 13$ would also be a valid solution since $13 \times 13 = 169$, $13 + 13 = 26$ and $169 + 26 = 195$.

If the problem asked for integer solutions, the number of solutions would increase still further. It is important to pay attention to any and all restrictions stated or implied in a problem.
Two cities, Mytown and Yourtown, had the same population at the end 2015. The population of Mytown decreased by 2.5% from the end of 2015 to the end of 2016. Then, the population increased by 8.4% from the end of 2016 to the end of 2017.

The population of Yourtown increased by $r\%$, $r > 0$, from the end of 2015 to the end of 2016. Then, the population of Yourtown increased by $(r + 2)\%$ from the end of 2016 to the end of 2017.

Surprisingly, the populations of both cities were the same again at the end of 2017. Determine the value of $r$ correct to one decimal place.
Problem of the Week
Problem E and Solution
A Tale of Two Cities

Problem

Two cities, Mytown and Yourtown, had the same population at the end 2015. The population of Mytown decreased by 2.5% from the end of 2015 to the end of 2016. Then, the population increased by 8.4% from the end of 2016 to the end of 2017. The population of Yourtown increased by \( r \% \), \( r > 0 \), from the end of 2015 to the end of 2016. Then, the population of Yourtown increased by \( (r + 2)\% \) from the end of 2016 to the end of 2017. Surprisingly, the populations of both cities were the same again at the end of 2017. Determine the value of \( r \) correct to one decimal place.

Solution

Let \( p \) be the population of Mytown at the end of 2015. Since Mytown and Yourtown have the same population size, then \( p \) is also the population of Yourtown at the end of 2015.

The population of Mytown decreased by 2.5% in 2016, so the population at the end of 2016 is

\[
p - \frac{2.5}{100} p = \left( 1 - \frac{2.5}{100} \right) p = 0.975p.
\]

The population of Mytown then increased by 8.4% during 2017, so the population at the end of 2017 is

\[
0.975p + \left( \frac{8.4}{100} \right) (0.975p) = \left( 1 + \frac{8.4}{100} \right) (0.975p) = 1.084(0.975p) = 1.0569p.
\]

The population of Yourtown increased by \( r\% \) in 2016, so the population at the end of 2016 is

\[
p + \frac{r}{100} p = \left( 1 + \frac{r}{100} \right) p.
\]

The population of Yourtown then increased by \( (r + 2)\% \) during 2017, so the population at the end of 2017 is

\[
\left( 1 + \frac{r}{100} \right) p + \frac{r + 2}{100} \left( 1 + \frac{r}{100} \right) p = \left( 1 + \frac{r}{100} \right) p \left( 1 + \frac{r + 2}{100} \right).
\]

Since the populations of Mytown and Yourtown are equal at the end of 2017, we have

\[
\left( 1 + \frac{r}{100} \right) \left( 1 + \frac{r + 2}{100} \right) p = 1.0569p
\]

\[
\left( \frac{100 + r}{100} \right) \left( \frac{100 + r + 2}{100} \right) = 1.0569, \quad \text{dividing both sides by } p, \text{ since } p > 0
\]

\[
\frac{100 + r}{100} \frac{102 + r}{100} = 10.569, \quad \text{multiplying both sides by 10 000 to clear fractions}
\]

\[
10 200 + 202r + r^2 = 10 569
\]

\[
r^2 + 202r - 369 = 0
\]

After using the quadratic formula and ruling out an inadmissible \( r \) value, we obtain \( r = 1.8\% \), correct to one decimal place.
Problem of the Week

Problem E

Paint Free

A cube has edges of length $n$, where $n$ is a positive integer. Three faces, meeting at a corner, are painted red. The cube is then cut into $n^3$ smaller cubes of unit length. That is, the side lengths of the new cubes are 1 unit.

If exactly 125 of these cubes have no faces painted red, determine the value of $n$. 
Problem of the Week
Problem E and Solution
Paint Free

Problem
A cube has edges of length \( n \), where \( n \) is a positive integer. Three faces, meeting at a corner, are painted red. The cube is then cut into \( n^3 \) smaller cubes of unit length. That is, the side lengths of the new cubes are 1 unit. If exactly 125 of these cubes have no faces painted red, determine the value of \( n \).

![Diagram of a cube with labels A, B, and C]

Solution
Solution 1
This solution requires no content beyond grade ten. The second solution will use the factor theorem which is generally taught in grade twelve.

In the diagram above, the sides painted red are labelled \( A \), \( B \), and \( C \). We know that there are \( n^3 \) unit cubes. To determine the number of unpainted cubes we can subtract the number of cubes with some red from the total number of cubes. Side \( A \) has dimensions \( n \) by \( n \) by 1 and so contains \( n^2 \) unit cubes with some red. Side \( B \) has dimensions \( n \) by \( (n - 1) \) by 1 and so contains \( n \times (n - 1) \) unit cubes with some red. Side \( C \) has dimensions \( (n - 1) \) by \( (n - 1) \) by 1 and so contains \( (n - 1)(n - 1) \) unit cubes with some red. The number of unpainted cubes is \( n^3 - n^2 - n(n - 1) - (n - 1)(n - 1) \). We can simplify this as follows:

\[
n^3 - n^2 - n(n - 1) - (n - 1)(n - 1) = n^2(n - 1) - n(n - 1) - (n - 1)(n - 1)
\]

Each term contains a common factor of \( (n - 1) \) so the expression simplifies to \( (n - 1)(n^2 - n - (n - 1)) = (n - 1)(n^2 - 2n + 1) \). This further simplifies to \( (n - 1)^3 \). If the solver pauses here to think about this, if the unit cubes on side \( A \) then side \( B \) and finally side \( C \) are removed we are left with a cube whose side lengths are \( (n - 1) \) and \( (n - 1)^3 \) unit cubes.

But \( (n - 1)^3 = 125 \), the actual number of unpainted cubes. Taking the cube root, \( n - 1 = 5 \) and \( n = 6 \) follows.
Solution 2

The second solution will use the factor theorem which is generally taught in grade twelve.

This solution picks up from the expression giving us the number of unpainted cubes, setting it equal to 125, the number of unpainted cubes.

\[
\begin{align*}
n^3 - n^2 - n(n - 1) - (n - 1)(n - 1) & = 125 \\
n^3 - n^2 - n^2 + n - n^2 + 2n - 1 & = 125 \\
n^3 - 3n^2 + 3n - 126 & = 0
\end{align*}
\]

Let \( f(n) = n^3 - 3n^2 + 3n - 126. \)

When \( n = 6, \) \( f(6) = 6^3 - 3(6^2) + 3(6) - 126 = 216 - 108 + 18 - 126 = 224 - 224 = 0. \) Since \( f(6) = 0, \) \((n - 6)\) is a factor of \( f(n).\)

After long division (or synthetic division), \( f(n) = (n - 6)(n^2 + 3n + 21). \)

So \((n - 6)(n^2 + 3n + 21) = 0.\) \(n^2 + 3n + 21 = 0\) has no real roots so \( n = 6 \) is the only root.

Therefore the original cube has edges of length 6.
In a survey, Grade 12 students were asked if they like apples. They were then asked if they like bananas. The information is summarized below.

- 30% of the students do not like apples
- 36 students do not like bananas.
- 60 students like both
- 48 students like one but not the other

How many students do not like apples and do not like bananas?
Problem of the Week

Problem E and Solution

Apples and Bananas

Problem
In a survey, Grade 12 students were asked if they like apples. They were then asked if they like bananas. The information is summarized: 30% of the students do not like apples; 36 students do not like bananas; 60 students like both; and 48 students like one but not the other. How many students do not like apples and do not like bananas?

Solution
We will set up a Venn Diagram.

Let \( z \) be the number of students that do not like apples and do not like bananas, \( x \) be the number of students that like only apples and \( y \) be the number of students that like only bananas. The 60 is the number of students who like both apples and bananas.

The number of students that do not like bananas is equal to the number of students outside the circle that is labelled bananas. This is \( x + z \). Therefore \( x + z = 36 \). Call this equation (1).

The number of students who like one but not the other is \( x + y = 48 \). Call this equation (2).

The total number of students is \( x + y + z + 60 \) and 30% of this is \( 0.3(x + y + z + 60) \). This is equal to the number of students that do not like apples, \( y + z \). Therefore, \( 0.3(x + y + z + 60) = y + z \). Call this equation (3).

Subtracting (1) from (2), we get \( y - z = 12 \). Therefore, \( y = z + 12 \). Call this equation (4).

Substituting (1) into (3) we get \( 0.3(36 + y + 60) = y + z \), or \( 0.3(96 + y) = y + z \). Now substitute equation (4) for \( y \),

\[
0.3(96 + (z + 12)) = (z + 12) + z
\]

\[
28.8 + 0.3z + 3.6 = 2z + 12
\]

\[
20.4 = 1.7z
\]

\[
z = 12
\]

Therefore, there are 12 students who do not like apples and do not like bananas.
Points $A$, $B$ and $D$ lie on the circumference of a circle with centre $C$. 
$\angle CAD = p^\circ$ and $\angle CBD = q^\circ$. Determine the measure of $\angle ACB$ and the measure of $\angle ADB$. What is the relationship between $\angle ACB$ and $\angle ADB$?
Problem of the Week
Problem E and Solution
Can You Relate?

Problem
Points $A$, $B$ and $D$ lie on the circumference of a circle with centre $C$. $\angle CAD = p^\circ$ and $\angle CBD = q^\circ$. Determine the measure of $\angle ACB$ and the measure of $\angle ADB$. What is the relationship between $\angle ACB$ and $\angle ADB$?

Solution
We start by constructing radius $CD$.

$CA$ and $CD$ are both radii of the circle, so $CA = CD$. Then $\triangle CAD$ is isosceles and $\angle CDA = \angle CAD = p^\circ$. Since the angles in a triangle add to $180^\circ$, $\angle ACD = (180 - 2p)^\circ$.

$CB$ and $CD$ are both radii of the circle, so $CB = CD$. Then $\triangle CBD$ is isosceles and $\angle CDB = \angle CBD = q^\circ$. Since the angles in a triangle add to $180^\circ$, $\angle BCD = (180 - 2q)^\circ$.

We will now find the measure of $\angle ADB$ and of $\angle ACB$ in order to determine the relationship.

$$\angle ADB = \angle CDB - \angle CDA = (q - p)^\circ$$
$$\angle ACB = \angle ACD - \angle BCD = (180 - 2p)^\circ - (180 - 2q)^\circ = (2q - 2p)^\circ = 2 \times (q - p)^\circ = 2 \times \angle ADB$$

$\therefore \angle ACB$ is double the size of $\angle ADB$.

In general, the angle inscribed at the centre of a circle is twice the size of the angle inscribed at the circumference by the same chord. In the following diagram, $\angle ACB$ is inscribed at the centre of the circle by chord $AB$ and $\angle ADB$ is inscribed at the circumference by the same chord. Therefore, $\angle ACB = 2\angle ADB$. 
Angela and Barry share a piece of land. The ratio of the area of Angela’s portion of land to the area of Barry’s portion of land is 3 : 2. They each grow corn and peas on their piece of land. The entire piece of land is covered by corn and peas in the ratio 7 : 3. On Angela’s portion of the land, the ratio of corn to peas is 4 : 1. What is the ratio of corn to peas for Barry’s portion of land?
Problem

Angela and Barry share a piece of land. The ratio of the area of Angela’s portion of land to the area of Barry’s portion of land is $3 : 2$. They each grow corn and peas on their piece of land. The entire piece of land is covered by corn and peas in the ratio $7 : 3$. On Angela’s portion of the land, the ratio of corn to peas is $4 : 1$. What is the ratio of corn to peas for Barry’s portion of land?

Solution

Solution 1

Suppose that Angela and Barry share 100 hectares of land. (We may assume any convenient total area.)

Since the ratio of the area of Angela’s land to the area of Barry’s land is $3 : 2$, then Angela has $\frac{3}{5}$ of the 100 hectares, or 60 hectares. Barry has the remaining 40 hectares.

Since the entire piece of land is covered by corn and peas in the ratio $7 : 3$, then $\frac{7}{10}$ of the 100 hectares (that is, 70 hectares) is covered with corn and the remaining 30 hectares with peas.

On Angela’s land, the ratio of corn to peas is $4 : 1$, so $\frac{4}{5}$ of her 60 hectares, or 48 hectares, is covered with corn and the remaining 12 hectares with peas.

Since there are 70 hectares of corn in total, then Barry has $70 - 48 = 22$ hectares of corn. Since there are 30 hectares of peas in total, then Barry has $30 - 12 = 18$ hectares of peas. Therefore, the ratio of corn to peas on Barry’s land is $22 : 18 = 11 : 9$.

Solution 2

Suppose that the total combined area of land is $x$.

Since the ratio of the area of Angela’s land to the area of Barry’s land is $3 : 2$, then Angela has $\frac{3}{5}$ of the land, or $\frac{3}{5}x$, while Barry has the remaining $\frac{2}{5}x$.

Since the entire piece of land is covered by corn and peas in the ratio $7 : 3$, then $\frac{7}{10}x$ is covered with corn and the remaining $\frac{3}{10}x$ with peas.

On Angela’s land, the ratio of corn to peas is $4 : 1$ so $\frac{4}{5}$ of her $\frac{3}{5}x$, or $\frac{4}{5} \left( \frac{3}{5}x \right) = \frac{12}{25}x$, is covered with corn and the remaining $\frac{3}{5}x - \frac{12}{25}x = \frac{3}{25}x$ with peas.

Since the area of corn is $\frac{7}{10}x$ in total, then Barry’s area of corn is $\frac{7}{10}x - \frac{12}{25}x = \frac{11}{50}x$.

Since the area of peas is $\frac{3}{10}x$ in total, then Barry’s area of peas is $\frac{3}{10}x - \frac{3}{25}x = \frac{9}{50}x$.

Therefore, the ratio of corn to peas on Barry’s land is $\frac{11}{50}x : \frac{9}{50}x = 11 : 9$. 
Three real numbers, $a, b$ and $c$, have a sum of 158 and a product of 74088. Also, $b = ar$ and $c = ar^2$, for some real values of $r$. Find all ordered triples $(a, b, c)$. 

\[(a, b, c)\]
Problem of the Week
Problem E and Solution
Not As Easy As a, b, c

Problem
Three real numbers, $a$, $b$ and $c$, have a sum of 158 and a product of 74088. Also, $b = ar$ and $c = ar^2$, for some real values of $r$. Find all ordered triples $(a, b, c)$.

Solution
Since $b = ar$, $c = ar^2$ and $abc = 74088$, then $a(ar)(ar^2) = a^3r^3 = 74088$, or $(ar)^3 = 74088$, or $ar = 42$.
Therefore, since $b = ar$, we have $b = 42$.

Now, $a + b + c = 158$ becomes $a + 42 + c = 158$, or $a + c = 116$.

Since $b = ar$, then $42 = ar$, or $r = \frac{42}{a}$ (since the product of $a$, $b$ and $c$ is not zero, we know $a \neq 0$).

Therefore, $c = a \left( \frac{42}{a} \right)^2 = a \left( \frac{1764}{a^2} \right) = \frac{1764}{a}$.

Substituting $c = \frac{1764}{a}$ into $a + c = 116$, we have

\[
\begin{align*}
    a + \frac{1764}{a} &= 116 \\
    a^2 + 1764 &= 116a \\
    a^2 - 116a + 1764 &= 0 \\
    (a - 18)(a - 98) &= 0
\end{align*}
\]

Therefore, $a = 18$ or $a = 98$.

When $a = 18$, then $r = \frac{42}{18} = \frac{7}{3}$, and one ordered triple is $(18, 42, 98)$.
Indeed, we can check that $18 + 42 + 98 = 158$ and $(18)(42)(98) = 74088$.

When $a = 98$, then $r = \frac{36}{98} = \frac{3}{7}$, and one ordered triple is $(98, 42, 18)$.
Indeed, we can check that $98 + 42 + 18 = 158$ and $(98)(42)(18) = 74088$.

In conclusion there are two ordered triples that satisfy the conditions of the problem: $(18, 42, 98)$ and $(98, 42, 18)$. 
Problem of the Week
Problem E
What is Possible?

Four cards each have a different positive integer printed on one side. Also, one card is purple, one card is blue, one card is green, and one card is yellow. (For those without a colour printer, the colour of the card is printed below the card.)

The four cards are placed on a table so that the numbered side is face down. Students look at the numbers on exactly three of the cards and then state the sum.

One student looked under the purple, blue and green cards and stated that the sum was 14. A second student looked under the blue, green and yellow cards and stated that the sum was 18. Finally, a third student looked under the purple, green and yellow cards and stated that the sum was 19.

Determine all possibilities for the positive integers on the purple, blue, green and yellow cards.

PURPLE  BLUE  GREEN  YELLOW
Problem

Four cards each have a different positive integer printed on one side. Also, one card is purple, one card is blue, one card is green, and one card is yellow. (For those without a colour printer, the colour of the card is printed below the card.) The four cards are placed on a table so that the numbered side is face down. Students look at the numbers on exactly three of the cards and then state the sum. One student looked under the purple, blue and green cards and stated that the sum was 14. A second student looked under the blue, green and yellow cards and stated that the sum was 18. Finally, a third student looked under the purple, green and yellow cards and stated that the sum was 19. Determine all possibilities for the positive integers on the purple, blue, green and yellow cards.

Solution

Solution 1

Let $p$, $b$, $g$, and $y$ represent the positive integers written on the purple, blue, green and yellow cards, respectively. It is given that $p \neq b \neq g \neq y$.

It is also given that  
\begin{align*}
    p + b + g &= 14 \quad (1) \\
    b + g + y &= 18 \quad (2) \\
    p + g + y &= 19 \quad (3)
\end{align*}

(2) − (1) gives $y − p = 4$ or equivalently, $y = p + 4$ \quad (4)

(3) − (2) gives $p − b = 1$ or equivalently, $b = p − 1$ \quad (5)

Since $p$, $b$, $g$ and $y$ are all different positive integers, let’s look at the possible values of $p$ and calculate the values of $b$, $g$ and $y$ in each case:

<table>
<thead>
<tr>
<th>Purple Card</th>
<th>Blue Card</th>
<th>Green Card</th>
<th>Yellow Card</th>
<th>Possible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$b = p − 1$ \quad (5)</td>
<td>$g = 14 − p − b$ \quad (from (1))</td>
<td>$y = p + 4$ \quad (4)</td>
<td>Yes or No</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>13</td>
<td>5</td>
<td>No ($b \neq 0$)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>11</td>
<td>6</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>Yes</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>No ($p = g$)</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>Yes</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>1</td>
<td>11</td>
<td>Yes</td>
</tr>
</tbody>
</table>

We can stop here, because if $p > 7$ then $g < 0$, which is not valid.

Therefore, there are 5 different possibilities for the numbers on the purple, blue, green and yellow cards, respectively.

They are $(p, b, g, y) = (2, 1, 11, 6), (3, 2, 9, 7), (4, 3, 7, 8), (6, 5, 3, 10), (7, 6, 1, 11)$.
Solution 2

This solution is very similar to solution 1. The key difference is that, in this solution, we find an expression for each of \( b, g, \) and \( y \), in terms of \( p \). We then determine the smallest and largest possible values for \( p \). Based on these values we calculate the values of the other unknowns and determine the validity of each possibility.

Let \( p, b, g, \) and \( y \) represent the positive integers written on the purple, blue, green and yellow cards, respectively. It is given that \( p \neq b \neq g \neq y \).

It is also given that
\[
\begin{align*}
p + b + g &= 14 \quad (1) \\
b + g + y &= 18 \quad (2) \\
p + g + y &= 19 \quad (3)
\end{align*}
\]

\( (2) - (1) \) gives \( y - p = 4 \) and \( y = p + 4 \) follows. \( (4) \)

\( (3) - (2) \) gives \( p - b = 1 \) and \( b = p - 1 \) follows. \( (5) \)

Substitute for \( y \) from \( (4) \) and \( b \) from \( (5) \) into \( (2) \) to get \( g \) in terms of \( p \).
\[
\begin{align*}
b + g + y &= 18 \\
(p - 1) + g + (p + 4) &= 18 \\
2p + g + 3 &= 18 \\
g &= 15 - 2p
\end{align*}
\]

Since \( b = p - 1 \) and \( b \) is a positive integer, the smallest positive integer value for \( p \) will be 2. Otherwise, \( b \leq 0 \).

Since \( g = 15 - 2p \) and \( g \) is a positive integer, the largest positive integer value for \( p \) will be 7. Otherwise, \( g \leq 0 \).

Therefore, the only possible values for \( p \) are \( \{2, 3, 4, 5, 6, 7\} \).

We will now look at each possible value of \( p \) and calculate the values of \( b, g \) and \( y \) in each case:

<table>
<thead>
<tr>
<th>Purple Card</th>
<th>Blue Card</th>
<th>Green Card</th>
<th>Yellow Card</th>
<th>Possible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( b = p - 1 )</td>
<td>( g = 15 - 2p )</td>
<td>( y = p + 4 )</td>
<td>Yes or No</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>11</td>
<td>6</td>
<td>Yes, ( p \neq b \neq g \neq y ).</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>Yes, ( p \neq b \neq g \neq y ).</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>Yes, ( p \neq b \neq g \neq y ).</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>No (( p = g ))</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>Yes, ( p \neq b \neq g \neq y ).</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>1</td>
<td>11</td>
<td>Yes, ( p \neq b \neq g \neq y ).</td>
</tr>
</tbody>
</table>

Therefore, there are 5 different possibilities for the numbers on the purple, blue, green and yellow cards, respectively.

They are \( (p, b, g, y) = (2, 1, 11, 6), (3, 2, 9, 7), (4, 3, 7, 8), (6, 5, 3, 10), (7, 6, 1, 11) \).
Problem of the Week
Problem E
Group of Six

$ABC$ is a three-digit integer whose first digit is $A$, second digit is $B$, and third digit is $C$. Similarly, $DEF$ is a three-digit integer whose first digit is $D$, second digit is $E$, and third digit is $F$.

We are given that

$$
\begin{array}{c}
A & B & C \\
+ & D & E & F \\
\hline
1 & 2 & 3 & 4
\end{array}
$$

How many 6-tuples $(A, B, C, D, E, F)$ are there with $A > D$, $B > E$, and $C > F$ that make the above statement true?

$$
(8, 9, 3, 3, 4, 1)
$$
Problem of the Week
Problem E and Solution
Group of Six

Problem

\(ABC\) is a three-digit integer whose first digit is \(A\), second digit is \(B\), and third digit is \(C\). Similarly, \(DEF\) is a three-digit integer whose first digit is \(D\), second digit is \(E\), and third digit is \(F\). We are given that

\[
\begin{array}{c}
A B C \\
+ D E F \\
\hline
1 2 3 4
\end{array}
\]

How many 6-tuples \((A, B, C, D, E, F)\) are there with \(A > D\), \(B > E\), and \(C > F\) that make the above statement true?

Solution

To solve this problem, we are going to look at each column starting with the units, then tens, and then finally the hundreds column.

Since \(C + F\) ends in a 4, then \(C + F = 4\) or \(C + F = 14\). The value of \(C + F\) cannot be 20 or more, as \(C\) and \(F\) are digits. In the case that \(C + F = 14\), we “carry” a 1 to the tens column.

Since the result in the tens column is 3, then when there is no “carry” from the units column, \(B + E\) ends in a 3, and when there is a “carry” from the units column \(1 + B + E\) ends in a 3, so \(B + E\) ends in a 2.

If \(B + E\) ends in a 3, then \(B + E = 3\) or \(B + E = 13\). The value of \(B + E\) cannot be 20 or more, as \(B\) and \(E\) are digits. In the case that \(B + E = 13\), we “carry” a 1 to the hundreds column.

If \(B + E\) ends in a 2, then \(B + E = 2\) or \(B + E = 12\). The value of \(B + E\) cannot be 20 or more, as \(B\) and \(E\) are digits. In the case that \(B + E = 12\), we “carry” a 1 to the hundreds column.

Since the result in the hundreds column is 12, then \(A + D = 12\), or in the case when there was a “carry” from the tens column \(1 + A + D = 12\), so \(A + D = 11\).

We summarize this information in a tree.

```
Possible Sums

C + F = 4
  / | \
B + E = 3  B + E = 13
  |     |     |
A + D = 12  A + D = 11

C + F = 14
  / | \
B + E = 2  B + E = 12
  |     |     |
A + D = 12  A + D = 11
```
We now look at the different possibilities for digits $A, B, C, D, E,$ and $F$ for each individual sum, with the restriction that $A > D$, $B > E$, and $C > F$.

<table>
<thead>
<tr>
<th>Sum</th>
<th>Possible solutions</th>
<th>Total Number of Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C + F = 4$</td>
<td>$C = 4, F = 0$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$C = 3, F = 1$</td>
<td></td>
</tr>
<tr>
<td>$C + F = 14$</td>
<td>$C = 9, F = 5$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$C = 8, F = 6$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 2$</td>
<td>$B = 2, E = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$B + E = 3$</td>
<td>$B = 3, E = 0$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$B = 2, E = 1$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 12$</td>
<td>$B = 9, E = 3$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$B = 8, E = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$B = 7, E = 5$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 13$</td>
<td>$B = 9, E = 4$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$B = 8, E = 5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$B = 7, E = 6$</td>
<td></td>
</tr>
<tr>
<td>$A + D = 11$</td>
<td>$A = 9, D = 2$</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$A = 8, D = 3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 7, D = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 6, D = 5$</td>
<td></td>
</tr>
<tr>
<td>$A + D = 12$</td>
<td>$A = 9, D = 3$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$A = 8, D = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 7, D = 5$</td>
<td></td>
</tr>
</tbody>
</table>

We can find the number of 6-tuples that are possible for the first branch of the tree. We have 2 choices for the units column. For each of these two choices, we have 2 choices for the tens column, so there are $2 \times 2 = 4$ possibilities. For each of these possibilities, we have 3 choices for the hundreds column. So the number of 6-tuples is $2 \times 2 \times 3 = 12$.

Similarly, for the second branch, we have 2 choices for the units column, 3 choices for the tens column, and 4 choices for the hundreds column. So the number of 6-tuples is $2 \times 3 \times 4 = 24$.

For the third branch, we have 2 choices for the units column, 1 choice for the tens column, and 3 choices for the hundreds column. So the number of 6-tuples is $2 \times 1 \times 3 = 6$.

For the fourth branch, we have 2 choices for the units column, 3 choices for the tens column, and 4 choices for the hundreds column. So the number of 6-tuples is $2 \times 3 \times 4 = 24$.

Therefore, the total number of 6-tuples is $12 + 24 + 6 + 24 = 66$. 
Problem of the Week
Problem E
Return Safely

An aircraft carrier is traveling at an average speed of 34 km/h on a course that is 60° west of south. A helicopter leaves the carrier and travels due north at an average speed of 200 km/h. The carrier maintains its course and average speed. The helicopter has enough fuel for five hours of flying and it maintains its constant speed of 200 km/h.

What is the maximum distance north the helicopter can travel, so that the fuel remaining will allow a safe return to the aircraft carrier?
Problem of the Week
Problem E and Solution
Return Safely

Problem
An aircraft carrier is traveling at an average speed of 34 km/h on a course that is 60° west of south. A helicopter leaves the carrier and travels due north at an average speed of 200 km/h. The carrier maintains its course and average speed. The helicopter has enough fuel for five hours of flying and it maintains its constant speed of 200 km/h.

What is the maximum distance north the helicopter can travel, so that the fuel remaining will allow a safe return to the aircraft carrier?

Solution
Solution 1
We know that the total time available is 5 hours and in that time the aircraft carrier will travel $5 \times 34 = 170$ km.

Let $t$ represent the time, in hours, that the helicopter travels due north. Then, the helicopter will have $(5 - t)$ hours to return to the carrier. The helicopter will fly $200t$ km due north and $200(5 - t) = 1000 - 200t$ km in a direction which allows the helicopter to return to the carrier.

On the diagram, $O$ represents the point at which the helicopter leaves the carrier, $A$ represents the farthest point north that the helicopter reaches and $B$ represents the point that the aircraft carrier reaches in 5 hours. $\angle AOB = 180° - 60° = 120°$. So $OA = 200t$ km, $OB = 170$ km, and $AB = (1000 - 200t)$ km. If we let $d = 200t$, we can simplify $OA$ to $d$ and $AB$ to $(1000 - d)$.

Using the cosine law,

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB) \cos(\angle AOB)$$

$$(1000 - d)^2 = d^2 + 170^2 - 2(170)(d) \cos(120°)$$

$1000000 - 2000d + d^2 = d^2 + 28900 - 340d \left( -\frac{1}{2} \right)$$

$1000000 - 2000d = 28900 + 170d$

$-2170d = -971100$

$d = \frac{971100}{2170}$

$d = 447.51$ km

The helicopter can travel about 447 km north before it must return to the final location of the carrier. Since $d = 200t$, we have $t = d \div 200 = 2$ h 14 min. (Both the distance and the time have been rounded down to allow for a safe return.) In terms of time, the pilot can fly north for about 2 hours and 14 minutes before having to return to the aircraft carrier.
Solution 2

Let \( t \) represent the maximum time that the helicopter can travel north. Let \( O \) represent the point where the helicopter leaves the carrier. Let \( A \) represent the most northerly point to which the helicopter could fly to before it must return to the carrier. Let \( B \) represent the location of the carrier after it travels five hours.

As in solution 1, we determine that the carrier travels \( 5 \times 34 = 170 \) km in five hours. We will describe the positions of the helicopter and the aircraft carrier in terms of coordinates. We know that the carrier is traveling through the third quadrant at an angle of \( 60^\circ \) from the negative \( y \)-axis and \( 30^\circ \) from the negative \( x \)-axis. The standard position angle related to these two angles would be \( 180^\circ + 30^\circ = 210^\circ \). We can determine the coordinates of \( B \), the location of the carrier after traveling 5 hours, using

\[
x = r \cos \theta = 170 \cos 210^\circ = 170 \left( -\frac{\sqrt{3}}{2} \right) = -85\sqrt{3}
\]

\[
y = r \sin \theta = 170 \sin 210^\circ = 170 \left( -\frac{1}{2} \right) = -85
\]

The coordinates of \( B \) are \((-85\sqrt{3}, -85)\).

Since the helicopter is traveling due north at 200 km/h for \( t \) hours, it travels \( 200t \) km and the coordinates of \( A \) are \((0, 200t)\). We can now calculate the distance from \( A(0, 200t) \) to \( B(-85\sqrt{3}, -85) \).

\[
(AB)^2 = (0 - (-85\sqrt{3}))^2 + (200t - (-85))^2
\]

\[
= (85\sqrt{3})^2 + (200t + 85)^2
\]

\[
= 7225(3) + 40000t^2 + 34000t + 7225
\]

\[
= 21675 + 40000t^2 + 34000t + 7225
\]

\[
= 40000t^2 + 34000t + 28900
\]

\[
= 100(400t^2 + 340t + 289)
\]

\[
AB = 10\sqrt{400t^2 + 340t + 289}, \quad AB > 0
\]

The time from \( A \) to \( B \) is

\[
t + \frac{\sqrt{400t^2 + 340t + 289}}{200} = \frac{\sqrt{400t^2 + 340t + 289}}{20}
\]

We already know that the time from \( O \) to \( A \) is \( t \) and that the total time is 5 hours. Then,

\[
t + \frac{\sqrt{400t^2 + 340t + 289}}{20} = 5
\]

\[
20t + \sqrt{400t^2 + 340t + 289} = 100
\]

\[
\sqrt{400t^2 + 340t + 289} = 100 - 20t
\]

Squaring both sides,

\[
400t^2 + 340t + 289 = 10000 - 4000t + 400t^2
\]

\[
4340t = 9711
\]

\[
t = \frac{9711}{4340}
\]

The maximum distance the helicopter can travel north is \( 200t = \frac{971100}{2170} \approx 447.51 \) km and the helicopter should not go north for more than 2 hours and 14 minutes.
Problem of the Week
Problem E
Only Four?

A three-digit number, \(abc\), is formed so that each of the digits in the number, \(a\), \(b\), and \(c\), are different. Four of these three-digit numbers have the property that the product of the digits of the original number is the same as the two-digit number formed by the tens digit and units digit of the original number, namely \(bc\).

One of the four numbers \(abc\) is 236, since \(a \times b \times c = 2 \times 3 \times 6 = 36\), which is \(bc\).

\[
2 \times 3 \times 6 = 36
\]

Find the remaining three numbers.

It may be helpful to recall that any two-digit number of the form \(bc\) can be represented by the sum \(10b + c\). For example, \(12 = 10(1) + 2\).
Problem of the Week
Problem E and Solution
Only Four?

Problem
A three-digit number, \( abc \), is formed so that each of the digits in the number, \( a, b, \) and \( c \), are different. Four of these three-digit numbers have the property that the product of the digits of the original number is the same as the two-digit number formed by the tens digit and units digit of the original number, namely \( bc \).
One of the four numbers \( abc \) is 236, since \( a \times b \times c = 2 \times 3 \times 6 = 36 \), which is \( bc \).

\[
2 \times 3 \times 6 = 36
\]

Find the remaining three numbers.

Solution
We know that there are only 4 answers, so we could attempt a trial and error approach to find the remaining 3 numbers. We will present a more systematic approach.

First, notice that since \( a, b \) and \( c \) are digits and must, therefore, be positive integers between 0 and 9. Also, since the product \( a \times b \times c \) is a two-digit number, none of \( a, b \) or \( c \) can equal zero.

We are asked to find all three-digit numbers \( abc \) such that \( a \times b \times c = 10b + c \).
Since \( b \neq 0 \), we can divide by \( b \) and the problem becomes equivalent to finding all integers \( a, b, c \) with \( 1 \leq a, b, c \leq 9 \) and \( a, b \) and \( c \) distinct such that

\[
a \times c = 10 + \frac{c}{b}
\]

Since \( a \) and \( c \) are integers, then so is \( a \times c \) and we must have that \( \frac{c}{b} \) is an integer as well. Therefore, for each possible value of \( c \), we must have that \( b \) divides exactly into \( c \).

We will break the problem into cases based on the value of \( c \) and then sub-cases based on what the value of \( b \) can be for that particular \( c \).

Case 1: \( c = 1 \)
There are no values of \( b \) where \( \frac{c}{b} \) is an integer and \( b \neq c \).

Case 2: \( c = 2 \)
For \( \frac{c}{b} \) to be an integer and \( b \neq c \), we must have \( b = 1 \). If \( b = 1 \) and \( c = 2 \), \( a \times b \times c = 10b + c \) becomes \( 2a = 12 \) and so \( a = 6 \). Therefore, one of the three-digit numbers is 612.
Case 3: $c = 3$

For $\frac{a \times b \times c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$. If $b = 1$ and $c = 3$, $a \times b \times c = 10b + c$ becomes $3a = 13$ and so $a = \frac{13}{3}$. Since $a$ is not an integer, there is no solution in this case.

Case 4: $c = 4$

For $\frac{a \times b \times c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$ or $b = 2$.

Case i: $b = 1$

In this case, $a \times b \times c = 10b + c$ becomes $4a = 14$ and so $a = \frac{7}{2}$. Since $a$ is not an integer, there is no solution in this case.

Case ii: $b = 2$

In this case, $a \times b \times c = 10b + c$ becomes $8a = 24$ and so $a = 3$. Therefore, one of the three-digit numbers is 324.

Case 5: $c = 5$

For $\frac{a \times b \times c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$. If $b = 1$ and $c = 5$, $a \times b \times c = 10b + c$ becomes $5a = 15$ and so $a = 3$. Therefore, one of the three-digit numbers is 315.

Case 6: $c = 6$

For $\frac{a \times b \times c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$, $b = 2$ or $b = 3$.

Case i: $b = 1$

In this case, $a \times b \times c = 10b + c$ becomes $6a = 16$ and so $a = \frac{8}{3}$. Since $a$ is not an integer, there is no solution in this case.

Case ii: $b = 2$

In this case, $a \times b \times c = 10b + c$ becomes $12a = 26$ and so $a = \frac{13}{6}$. Since $a$ is not an integer, there is no solution in this case.

Case iii: $b = 3$

In this case, $a \times b \times c = 10b + c$ becomes $18a = 36$ and so $a = 2$. Therefore, one of the three-digit numbers is 236. This is the number given in the example.

We can actually stop here since we have found 4 different three-digit numbers that satisfy the conditions outlined in the problem. If we had not been given the number of possible solutions, we would need to continue by checking cases when $c = 7$, $c = 8$ and $c = 9$.

Therefore, the 4 three-digit numbers that satisfy the conditions of the problem are 612, 324, 315, and 236.
Problem of the Week
Problem E
Boxed In

*CEMC Parcel* is a company that ships their product in boxes with certain restrictions. The company will only ship their product in boxes where

- the length, width, and height, in cm, are all integers,
- the length, width and height are in the ratio $4 : 3 : 5$,
- the sum of the length, width and height is at least 100 cm and at most 1000 cm,
- the volume of the box is less than $2 \, \text{m}^3$.

Determine the dimensions of the smallest box and of the largest box that satisfy *CEMC Parcel’s* restrictions.
Problem of the Week
Problem E and Solution
Boxed In

Problem
CEMC Parcel is a company that ships their product in boxes with certain restrictions. The company will only ship their product in boxes where

- the length, width, and height, in cm, are all integers,
- the length, width and height are in the ratio $4 : 3 : 5$,
- the sum of the length, width and height is at least 100 cm and at most 1000 cm,
- the volume of the box is less than $2 \text{ m}^3$.

Determine the dimensions of the smallest box and of the largest box that satisfy CEMC Parcel’s restrictions.

Solution
Since the boxes used by CEMC Parcel have integer side lengths in the ratio $4 : 3 : 5$, let $4n$ represent the length of a box in cm, let $3n$ represent the width of a box in cm, and let $5n$ represent the height of a box in cm, where $n$ is an integer.

Furthermore, the sum of the length, width and height must be at least 100 cm and at most 1000 cm. So it follows that

$$4n + 3n + 5n \geq 100 \quad \text{and} \quad 4n + 3n + 5n \leq 1000$$

$$12n \geq 100 \quad \quad \quad \quad \quad \quad \quad \quad \quad 12n \leq 1000$$

$$n \geq \frac{100}{12} \quad \quad \quad \quad \quad \quad \quad \quad \quad n \leq \frac{1000}{12}$$

$$n \geq \frac{81}{3} \quad \quad \quad \quad \quad \quad \quad \quad \quad n \leq 32.2$$

There is one other restriction to consider. The volume of a box used by CEMC Parcel is less than $2 \text{ m}^3$. To convert from m$^3$ to cm$^3$, note that

$$1 \text{ m}^3 = 1 \text{ m} \times 1 \text{ m} \times 1 \text{ m} = 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm} = 1 000 000 \text{ cm}^3$$

so $2 \text{ m}^3 = 2 000 000 \text{ cm}^3$.

Therefore,

$$4n(3n)(5n) < 2 000 000$$

$$60n^3 < 2 000 000$$

$$n^3 < \frac{100 000}{3}$$

$$n < \sqrt[3]{\frac{100 000}{3}} \approx 32.2$$

We also know that $n$ is an integer. Since $n \geq 8\frac{1}{3}$, then the smallest possible integer value of $n$ is 9. Since $n \leq 83\frac{1}{3}$ and $n < 32.2$, then the largest possible value of $n$ is 32.

Using the dimensions $4n$, $3n$ and $5n$ with $n = 9$, the smallest box has dimensions 36 cm by 27 cm by 45 cm. Using the dimensions $4n$, $3n$ and $5n$ with $n = 32$, the largest box has dimensions 128 cm by 96 cm by 160 cm. (The largest box has volume approximately 1.97 m$^3$.)
Counting, Probability & Data Analysis
The mean, median and mode of the eight positive numbers 10, 2, 5, 2, 6, 4, 2, x are distinct. The mean, median and mode are calculated and then listed in order from smallest to largest. The differences between adjacent numbers in this new list are equal.

Determine all possible values of x.
Problem of the Week

Problem E and Solution
Arithmetic Tendencies

Problem

The mean, median and mode of the eight positive numbers 10, 2, 5, 2, 6, 4, 2, \(x\) are distinct. The mean, median and mode are calculated and then listed in order from smallest to largest. The differences between adjacent numbers in this new list are equal. Determine all possible values of \(x\).

Solution

Since there are at least three 2’s, the mode will be 2 regardless the value of \(x\).

The mean of the numbers is \[
\frac{10 + 2 + 5 + 2 + 6 + 4 + 2 + x}{8} = \frac{x + 31}{8}.
\]

The median of the numbers will depend on the value of \(x\) compared to the other numbers.

Since there are eight numbers in the list, the median will be the average of the fourth and fifth numbers in the ordered list of numbers. We will break the problem into cases.

Case 1: 0 < \(x\) ≤ 2 and the ordered list is \(2, 2, 2, 2, 4, 5, 6, 10\).

The median is the average of the fourth and fifth numbers, or \(\frac{4 + 5}{2} = 3\).

Since \(x > 0\), the mean is \(\frac{x + 31}{8} > \frac{31}{8} > 3\). It follows that the mean is greater than the median. The ordered list of numbers is then \(2, 3, \frac{x + 31}{8}\).

Since the differences between adjacent numbers are equal, we have
\[
\begin{align*}
3 - 2 &= \frac{x + 31}{8} - 3 \\
1 &= \frac{x + 31}{8} - 3 \\
4 &= \frac{x + 31}{8} \\
32 &= x + 31 \\
1 &= x
\end{align*}
\]

So \(x = 1\) and the mode, median and mean are 2, 3, 4, respectively. Each adjacent pair of numbers in the list differs by 1.

Case 2: 2 < \(x\) ≤ 5, and the ordered list is \(2, 2, 2, x, 4, 5, 6, 10\) or \(2, 2, 2, 4, x, 5, 6, 10\).

In both lists, the fourth and fifth numbers are \(x\) and 4. It follows that the median for both lists is \(\frac{x + 4}{2}\). We also know that the mode is 2. We do not, however, know which is larger, the median or the mean, so we look at both cases.
Case 2a: Let the median be smaller than the mean. Then the ordered list is

\[2, \frac{x + 4}{2}, \frac{x + 31}{8}\]

Since the differences between adjacent numbers are equal, we have

\[
\frac{x + 4}{2} - 2 = \frac{x + 31}{8} - \frac{x + 4}{2}
\]

\[4x + 16 - 16 = x + 31 - 4x - 16\]  \(\text{(Multiply both sides of the equation by 8.)}\)

\[7x = 15\]
\[x = \frac{15}{7}\]

When \(x = \frac{15}{7}\) the mode is 2, the median becomes \(\frac{15}{7} + 4 = \frac{43}{14}\) and the mean becomes \(\frac{15}{7} + 31\) \(\frac{8}{7} = \frac{29}{7}\). Each adjacent pair of numbers in the list differs by \(\frac{15}{14}\).

Case 2b: Let the mean be smaller than the median. Then the ordered list is

\[2, \frac{x + 31}{8}, \frac{x + 4}{2}\]

Since the differences between adjacent numbers are equal, we have

\[
\frac{x + 31}{8} - 2 = \frac{x + 4}{2} - \frac{x + 31}{8}
\]

\[x + 31 - 16 = 4x + 16 - x - 31\]  \(\text{(Multiply both sides of the equation by 8.)}\)

\[30 = 2x\]
\[x = 15\]

But \(x \leq 5\), so there is no value of \(x\) that satisfies the conditions in this case.

Case 3: \(x > 5\) and the first 5 numbers in the list, in numerical order, are 2, 2, 2, 4, 5.

Since the fourth number in the list is 4 and the fifth number is 5, then it follows that the median is \(\frac{9}{2}\). The mode is 2. Since \(x > 5\), the mean is \(\frac{x + 31}{8} > \frac{36}{8} = \frac{9}{2}\). It follows that the mean is greater than the median. The ordered list of numbers is then \(2, \frac{9}{2}, \frac{x + 31}{8}\).

Since the differences between adjacent numbers are equal, we have

\[\frac{9}{2} - 2 = \frac{x + 31}{8} - \frac{9}{2}\]
\[\frac{7}{8} = \frac{x + 31}{8}\]
\[56 = x + 31\]
\[25 = x\]

In this case, \(x > 5\) and therefore \(x = 25\) is a valid solution.
When $x = 25$ the mode is 2, the median $\frac{9}{2}$ and the mean becomes $\frac{25 + 31}{8} = 7$. Each adjacent pair of numbers in the list differs by $\frac{5}{2}$.

Therefore, there are three values of $x$ that satisfy the conditions of the problem.

When $x = 1$, the ordered list is $1, 2, 2, 2, 4, 5, 6, 10$. The mean is 4, the median is 3 and the mode is 2. When listed from smallest to largest, the three numbers are 2, 3, 4 and the difference between adjacent terms is 1.

When $x = \frac{15}{7}$, the ordered list is $2, 2, 2, \frac{15}{7}, 4, 5, 6, 10$. The mean is $\frac{29}{7}$, the median is $\frac{43}{14}$ and the mode is 2. When listed from smallest to largest, the three numbers are $2, \frac{43}{14}, \frac{29}{7}$ and the difference between adjacent terms is $\frac{15}{14}$.

When $x = 25$, the ordered list is $2, 2, 2, 4, 5, 6, 10, 25$. The mean is 7, the median is $\frac{9}{2}$ and the mode is 2. When listed from smallest to largest, the three numbers are $2, \frac{9}{2}, 7$ and the difference between adjacent terms is $\frac{5}{2}$.

Note: 2, 3, 4 and $2, \frac{43}{14}, \frac{29}{7}$ and $2, \frac{9}{2}, 7$ are all examples of arithmetic sequences. An arithmetic sequence is a list of numbers with the difference between adjacent numbers being constant.
Problem of the Week
Problem E
Roll With It

A die, with the numbers 1, 2, 3, 4, 6, and 8 on its six faces, is rolled. (A net showing the six faces of the die is illustrated below.)

If, after the first roll, the number appearing on the top face of the die is odd, then all of the odd numbers on the die are doubled. If, after the first roll, the number appearing on the top face of the die is even, then all of the even numbers on the die are halved. This new die is rolled. The rules stated above are applied to the outcome of this roll producing another new die.

This second new die is then rolled. No change occurs after this roll. What is the probability that a 2 will be on the top face of the die after the third roll?
Problem of the Week
Problem E and Solution
Roll With It

Problem
A die, with the numbers 1, 2, 3, 4, 6, and 8 on its six faces, is rolled. (A net showing the six faces of the die is illustrated above.) If, after the first roll, the number appearing on the top face of the die is odd, then all of the odd numbers on the die are doubled. If, after the first roll, the number appearing on the top face of the die is even, then all of the even numbers on the die are halved. This new die is rolled. The rules stated above are applied to the outcome of this roll producing another new die. This second new die is then rolled. No change occurs after this roll. What is the probability that a 2 will be on the top face of the die after the third roll?

Solution
We will use the notation \((1, 2, 3, 4, 6, 8)\) to describe the original die.
For the first two rolls, we need to keep track of the parity only. For the third roll we need to look at an outcome of 2. There are therefore four possible cases for the first two rolls.

Case 1: First roll is even, second roll is even.
The probability of rolling an even on the first roll is \(\frac{4}{6} = \frac{2}{3}\). The die will now be \((1, 1, 3, 2, 3, 4)\).
The probability of rolling an even on the second roll is \(\frac{2}{6} = \frac{1}{3}\). The die will now be \((1, 1, 3, 1, 3, 2)\).
The probability of rolling a 2 on the third roll is \(\frac{1}{6}\).
The probability of rolling a 2 in this case is \(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} = \frac{1}{27}\).

Case 2: First roll is even, second roll is odd.
The probability of rolling an even on the first roll is \(\frac{2}{3}\). The die will now be \((1, 1, 3, 2, 3, 4)\).
The probability of rolling an odd on the second roll is \(\frac{2}{3}\). The die will now be \((2, 2, 6, 2, 6, 4)\).
The probability of rolling a 2 on the third roll is \(\frac{3}{6} = \frac{1}{2}\).
The probability of rolling a 2 in this case is \(\frac{2}{3} \times \frac{2}{3} \times \frac{1}{2} = \frac{4}{18}\).

Case 3: First roll is odd, second roll is even.
The probability of rolling an odd on the first roll is \(\frac{1}{3}\). The die will now be \((2, 2, 6, 4, 6, 8)\).
The probability of rolling an even on the second roll is 1. The die will now be \((1, 1, 3, 2, 3, 4)\).
The probability of rolling a 2 on the third roll is \(\frac{1}{6}\).
The probability of getting a 2 in this case is \(\frac{1}{3} \times 1 \times \frac{1}{6} = \frac{1}{18}\).

Case 4: First roll is odd, second roll is odd.
The probability of rolling an odd on the first roll is \(\frac{1}{3}\). The die will now be \((2, 2, 6, 4, 6, 8)\).
The probability to rolling an odd on the second roll is 0.
The probability of getting a 2 in this case is 0.

The total probability is the sum of the probabilities of each case or \(\frac{1}{27} + \frac{4}{18} + \frac{1}{18} + 0 = \frac{17}{54}\).
Therefore the probability of rolling a 2 on the third roll is \(\frac{17}{54}\).
Problem of the Week

Problem E

Llama Mall

The title, “Llama Mall”, is an example of a palindrome, a phrase that is the same when read forwards or backwards. Single words like MOM and BOB are palindromes. Numbers like 7, 414 and 12321 are also palindromes.

Next week, on October 8, 2018, you should greet everyone by saying, “Happy Palindrome Day”. When the date is written in the form d-mm-yyyy, October 8, 2018 is 8102018, a palindrome.

In honour of Palindrome Day, we pose a palindrome problem.

There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there?
Problem
The title, “Llama Mall”, is an example of a palindrome, a phrase that is the same when read forwards or backwards. Single words like MOM and BOB are palindromes. Numbers like 7, 414 and 12321 are also palindromes. There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there?

Solution
Our first observation is that since we are adding two four-digit palindromes to form a five-digit palindrome, then for some digits \( a, b, c, d, e, f, g \), we must have

\[
\begin{align*}
  a b b a \\
  + c d d c \\
  \hline
  e f g f e
\end{align*}
\]

Also, since this five-digit number is the sum of two four-digit numbers, \( e \) must be 1. So we now have,

\[
\begin{align*}
  a b b a \\
  + c d d c \\
  \hline
  1 f g f 1
\end{align*}
\]

From the units column we know that \( a + c \) has a units digit of 1. Also, since \( a \) and \( c \) are digits, then \( a + c < 20 \). It cannot be the case that \( a + c = 1 \), because then otherwise either \( a = 0 \) or \( c = 0 \) (and then we are not adding two four-digit palindromes). Thus, \( a + c = 11 \). We list the possibilities for \( a \) and \( c \):

<table>
<thead>
<tr>
<th>( a )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
</tr>
</tbody>
</table>

Note that there are only four possibilities here. If we extended this table, we would get an additional four possibilities which would be duplicates of these four, with \( a \) and \( c \) reversed.
Since \( a + c = 11 \), the first two digits of the sum are either 11 (if there is no carry from the hundreds column) or 12 (if there is a carry from the hundreds column). This means either \( f = 1 \) or \( f = 2 \). Let’s look at both options.

**Option 1: \( f = 1 \)**

\[
\begin{array}{cc}
  a & b \\
  + & c \\
  \hline
  1 & 1 \\
\end{array}
\]

Since \( f = 1 \), then there is no carry from the hundreds column. Since there is no carry, this means that \( b + d \) is a single digit, and from the tens digit of the sum we get \( b + d + 1 = 1 \). Thus \( b + d = 0 \), so \( b = 0 \) and \( d = 0 \). Notice that for each of the four possibilities for \( a \) and \( c \), selecting \( b = d = 0 \) will generate a valid pair of four-digit palindromes. Therefore, there are a total of 4 four-digit palindrome pairs where \( f = 1 \).

**Option 2: \( f = 2 \)**

\[
\begin{array}{cc}
  a & b \\
  + & c \\
  \hline
  1 & 2 \\
\end{array}
\]

Since \( f = 2 \), then there is a carry from the hundreds column. From the tens column, \( b + d + 1 \) must end in 2. Also, since there is a carry from the hundreds column, then \( b + d + 1 = 12 \), so \( b + d = 11 \). There are eight ways for this to happen:

\[
\begin{array}{cccccccc}
  b & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  d & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\
\end{array}
\]

Notice that provided \( a + c = 11 \), each of these 8 possibilities for \( b \) and \( d \) will produce a valid pair of four-digit palindromes. Therefore, for each of the four possibilities for \( a \) and \( c \), there are 8 ways to produce a valid four-digit palindrome pair. Therefore, there are a total of \( 4 \times 8 = 32 \) four-digit palindrome pairs where \( f = 2 \).

Therefore, the total number of pairs of four-digit palindromes that sum to a five-digit palindrome is \( 4 + 32 = 36 \).
Problem of the Week
Problem E
Going Bananas

A maple tree 🌴 is surrounded by two pine trees 🌲 and two palm trees 🌴, as follows:

Five types of bananas, $P, Q, R, S, T$ are placed in the trees. Each tree has exactly one type of banana in it and there is a different type of banana in each tree. A monkey is hungry and takes the same amount of time to eat any banana. The monkey starts on a tree and begins by eating a banana on that tree. The monkey then swings to another tree and eats a banana on that tree. The monkey then swings to another tree and eats a banana on that tree. This will continue until the monkey stops.

It takes the monkey

- three seconds to swing from the maple tree to any other tree or vice versa,
- two seconds to swing from a pine tree to a palm tree or vice versa, and
- seven seconds to swing between two pine trees or two palm trees while avoiding the maple tree along the way.

The monkey eats bananas of type $P, Q, S, R, T, R, P$ in that order then stops. List all the possible types of banana that can be in the maple tree if the total amount of time the monkey swings is as small as possible.
Problem of the Week
Problem E and Solution
Going Bananas

Problem

A maple tree is surrounded by two pine trees and two palm trees, as shown above. Five types of bananas, $P, Q, R, S, T$ are placed in the trees. Each tree has exactly one type of banana in it and there is a different type of banana in each tree. A monkey is hungry and takes the same amount of time to eat any banana. The monkey starts on a tree and begins by eating a banana on that tree. The monkey then swings to another tree and eats a banana on that tree. The monkey then swings to another tree and eats a banana on that tree. This will continue until the monkey stops. It takes the monkey three seconds to swing from the maple tree to any other tree or vice versa, two seconds to swing from a pine tree to a palm tree or vice versa, and seven seconds to swing between two pine trees or two palm trees while avoiding the maple tree along the way. The monkey eats bananas of type $P, Q, S, R, T, R, P$ in that order then stops. List all the possible types of banana can be in the maple tree if the total amount of time the monkey swings is as small as possible.

Solution

The given sequence $P, Q, S, R, T, R, P$ visits all the banana types with six swings $P - Q, Q - S, S - R, R - T, T - R, R - P$. For this sequence, the monkey swings to each tree at least once and from each tree at least once. Therefore, there needs to be a swing to and a swing from the maple. So the minimum time that is possible occurs when there are only two swings to and from the maple and the other four swings are between the pine and palm trees. Thus, the minimum possible time is $2(3) + 4(2) = 14$ seconds. We can show three such routes below.

On the next page we will show why it is not possible to have a 14 second route if the bananas on the maple tree are of type $R$ or $Q$. 
If the bananas on the maple are of type $R$ then there are four swings to or from the maple and the minimum time is now $4(3) + 2(2) = 16$. This is more than 14 seconds. Therefore, if the bananas on the maple are of type $R$, we cannot achieve the minimum time of 14 seconds.

If the bananas on the maple tree is are of type $Q$, then to obtain a total time of 14 seconds, the monkey must take two seconds to swing from each of $S$ to $R$, $R$ to $T$ and $R$ to $P$.

This means that the bananas $S, T,$ and $P$ are all palm trees or all pine trees, which contradicts the initial situation.

Another way of thinking about this is that since the monkey will swing from each of $S$ to $R$, $R$ to $T$ and $R$ to $P$ then at least one of the swings must take 7 s and the minimum time is $2(3) + 3(2) + 7 = 19$ seconds. This is more than 14 seconds. Therefore, if the bananas on the maple are of type $Q$, we cannot achieve the minimum time of 14 seconds.

In conclusion, the bananas on the maple tree could be type $P, S,$ or $T$.

**Applications to Computer Science**

This problem involves finding the best, or optimal, solution to a problem. Computers are often used to find the maximum or minimum value of some measurement. In this case, we might think of the trees as applications on a touch screen and the monkey swinging as the movement of a human finger from one application to another. A user interface designer might be interested in how to arrange the applications for a common sequence of operations so as to require as little time as possible.
In a game, players compete to earn points. The banker “pays” the players using tokens. A circular token is worth 1 point, a square token is worth 5 points, a triangular token is worth 10 points, and a hexagonal token is worth 25 points.

When paying a player, the banker must follow the minimum token rule. That is, the banker must pay using the least number of tokens possible. For example, to pay a player 30 points, there are many combinations of tokens that work. The banker could use 30 circular tokens \((30 \times 1 \text{ point} = 30 \text{ points})\) or 2 square tokens and 2 triangular tokens \((2 \times 5 \text{ points} + 2 \times 10 \text{ points} = 30 \text{ points})\), a total of four tokens. There are other possible ways to combine tokens to obtain 30 points. However, since the banker must pay using the minimum token rule, the banker would pay using 2 tokens, 1 hexagonal token and 1 square token \((1 \times 25 \text{ points} + 1 \times 5 \text{ points} = 30 \text{ points})\).

How many different point totals can the banker generate using exactly four tokens, provided the banker follows the minimum token rule? For example, a total of 41 points can be generated using exactly four tokens since this is the minimum number of tokens required to produce that total. A total of 30 points would not be generated using exactly four tokens since this total can be produced using fewer tokens.
Problem
In a game, players compete to earn points. The banker “pays” the players using tokens. A circular token is worth 1 point, a square token is worth 5 points, a triangular token is worth 10 points, and a hexagonal token is worth 25 points. When paying a player, the banker must follow the minimum token rule. That is, the banker must pay using the least number of tokens possible. For example, to pay a player 30 points, there are many combinations of tokens that work. The banker could use 30 circular tokens \((30 \times 1 \text{ point} = 30 \text{ points})\) or 2 square tokens and 2 triangular tokens \((2 \times 5 \text{ points} + 2 \times 10 \text{ points} = 30 \text{ points})\), a total of four tokens. There are other possible ways to combine tokens to obtain 30 points. However, since the banker must pay using the minimum token rule, the banker would pay using 2 tokens, 1 hexagonal token and 1 square token \((1 \times 25 \text{ points} + 1 \times 5 \text{ points} = 30 \text{ points})\). How many different point totals can the banker generate using exactly four tokens, provided the banker follows the minimum token rule?

Solution
For the solution, we will consider cases to carefully count the possibilities.

4 circular tokens
There is only one possibility here: four circular tokens gives a total of 4 points.

3 circular tokens
There are three possibilities here: three circular tokens and one square token giving a total of 8 points, three circular tokens and one triangular token giving a total of 13 points, and three circular tokens and one hexagonal token giving a total of 28 points.

2 circular tokens
• and two of one other token.
  This leads to two possibilities: two circular tokens and two triangular tokens for a total of 22 points, and two circular tokens and two hexagonal tokens for a total of 52 points. Note that two circular tokens and two square tokens gives a total of 12 points. This same total can be made using fewer tokens, namely two circular tokens and one triangular token, and is therefore not valid.
  • and one of each of two other tokens.
  This leads to three valid possibilities: two circular tokens, one square token and one triangular for a total of 17 points; two circular tokens, one square token and one hexagonal token for a total of 32 points; and two circular tokens, one triangular token and one hexagonal token for a total of 37 points.

1 circular token
• and three of one other type of token.
  This leads to one valid possibility: one circular token and three hexagonal tokens for a sum of 76 points. The possibilities one circular token and three square tokens, and one circular token and three triangular tokens are both invalid since the totals produced in each case can be made using fewer tokens. (This is left for the solver to verify.)
1 circular token (continued)

- and two of a second type of token and one of a third type of token.
  This leads to three valid possibilities: one circular token, one square token and two
  hexagonal tokens for a total of 56 points; one circular token, one triangular token and two
  hexagonal tokens for a total of 61 points; and one circular token, one hexagonal token and
  two triangular tokens for a total of 46 points. The totals obtained using one circular token,
  one square token and two triangular tokens, or one circular token, one triangular token and
  two square tokens, or one circular token, one hexagonal token and two square tokens can
  be obtained using fewer tokens and hence are invalid. (Again, this is left for the solver to
  verify.)

- and one of each of the other three tokens.
  This leads to one valid possibility: one circular token, one square token, one triangular
  token and one hexagonal token for a total of 41 points.

No circular tokens

- and four of one other token.
  There is only one valid possibility here: using four hexagonal tokens, we obtain a total of
  100 points. The other two possibilities made using 4 of the same token are both invalid
  since the totals can be made using fewer tokens. Four square tokens gives a total of
  20 points. The same total can be made using two triangular tokens and hence fewer tokens.
  Four triangular tokens gives a total of 40 points. The same total can be made using one
  hexagonal token, one triangular token and one square token, and hence fewer tokens.

- with three of a second type of token and one of a third type of token.
  This leads to two valid possibilities: three hexagonal tokens and one square token for a
  total of 80 points, and three hexagonal tokens and one triangular token for a total of
  85 points. Using three square tokens and any other token would be invalid since three
  square tokens could be replaced with two tokens, a triangular token and a square token, to
  produce the same total points. Using three triangular tokens and any other token would be
  invalid since three triangular tokens could be replaced with two tokens, a hexagonal token
  and a square token, to produce the same total points.

- with two of one type of token and two of another type of token.
  This leads to one valid possibility: two triangular tokens and two hexagonal tokens for a
  total of 70 points. Two square tokens could never be paired with any other pair of tokens
  since two square tokens can be replaced by one triangular token, and hence fewer tokens to
  produce the same total.

- two of one type of token and one of each of the remaining two different tokens.
  This leads to one valid possibility: two hexagonal tokens, one square token, and one
  triangular token for a total of 65 points. Two triangular tokens, one square token and one
  hexagonal token gives a total of 50 points. This total can be made with fewer tokens by
  using two hexagonal tokens. Two square tokens, one triangular token and one hexagonal
  token gives a total of 45 points. This total can be made with fewer tokens by using one
  hexagonal token and two triangular tokens.

The number of different totals is the obtained by adding the number of possibilities from each case,
namely \[ 1 + 3 + 2 + 3 + 1 + 3 + 1 + 1 + 2 + 1 + 1 = 19 \] possible totals.

The 18 possible totals using exactly four tokens are

\[ 4, 8, 13, 17, 22, 28, 32, 37, 41, 46, 52, 56, 61, 65, 70, 76, 80, 85, 100. \]
Problem of the Week

Problem E

Getting There

In the diagram, $O$ is the origin and $P$ is the point $(6, -4)$. Many paths exist that can get us from point $O$ to point $P$. Two such paths are shown on the grid.

We define the path length between two points $A$ and $B$ as the minimum length along the grid lines from $A$ to $B$.

The path length from $O$ to $P$ is 10. One such path is shown with solid line segments. A second path, shown with dashed lines, has length 20, which is greater than the minimum length path.

How many points with integer coordinates have a path length of 10 from $O$?
Problem of the Week
Problem E and Solution
Getting There

Problem
We define the *path length* between two points $A$ and $B$ as the minimum length along the grid lines from $A$ to $B$. In the diagram, $O$ is the origin and $P$ is the point $(6, -4)$. The path length from $O$ to $P$ is 10. How many points with integer coordinates have a path length of 10 from $O$?

Solution
Solution 1
Let $Q(a, b)$ be a point that has path length of 10 from $O$, the origin.

Let’s first assume that $Q$ is on the $x$ or $y$ axis.
The only point along the positive $x$-axis that has path length 10 from the origin is $(10, 0)$.
The only point along the negative $x$-axis that has path length 10 from the origin is $(-10, 0)$.
The only point along the positive $y$-axis that has path length 10 from the origin is $(0, 10)$.
The only point along the negative $y$-axis that has path length 10 from the origin is $(0, -10)$.
Therefore, there are 4 points along the axes that have a path length of 10 from $O$.

Next, let’s assume $a > 0$ and $b > 0$, so $Q$ is in the first quadrant.
Since the path length from $O$ to $Q$ is 10, there must be a path from $O$ to $Q$ that moves a total of $r$ units to the right and $u$ units up (in some order) such that $r + u = 10$. This means that $Q$ is $r$ units to the right of $O$ and $u$ units up from $O$. In other words, $a = r$ and $b = u$, so $a + b = r + u = 10$.

The points $(a, b)$ in the first quadrant that satisfy $a + b = 10$ where $a$ and $b$ are integers are $(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)$. There are 9 such pairs. Therefore, there are 9 points in the first quadrant that have path length of 10 from $O$.

By symmetry, there are 9 points in each of the four quadrants that have path length of 10 from $O$.
In quadrant 2, the points are $(-1, 9), (-2, 8), (-3, 7), (-4, 6), (-5, 5), (-6, 4), (-7, 3), (-8, 2), (-9, 1)$.
In quadrant 3, the points are $(-1, -9), (-2, -8), (-3, -7), (-4, -6), (-5, -5), (-6, -4), (-7, -3), (-8, -2), (-9, -1)$.
In quadrant 4, the points are $(1, -9), (2, -8), (3, -7), (4, -6), (5, -5), (6, -4), (7, -3), (8, -2), (9, -1)$.
Therefore, there are a total of $4 + (4 \times 9) = 40$ points with integer coordinates that have a path length of 10 from $O$. 

---
Solution 2

We are permitted ten moves to get from the origin to a point by traveling along the grid lines. These moves can be all horizontal (in one direction), all vertical (in one direction), or a combination of horizontal moves (in one direction) with vertical moves (in one direction).

We will examine the cases by picking the number of horizontal moves. Then we will determine the corresponding number of vertical moves and the resulting possible endpoints.

- **0 horizontal moves**: Since there are no horizontal moves, there would be 10 vertical moves. There are two possible endpoints, (0, 10), and (0, -10).

- **1 horizontal move**: Since there is 1 horizontal move, there would be 9 vertical moves. There are four possible endpoints, (-1, 9), (-1, -9), (1, 9), and (1, -9).

- **2 horizontal moves**: Since there are 2 horizontal moves, there would be 8 vertical moves. There are four possible endpoints, (-2, 8), (-2, -8), (2, 8), and (2, -8).

- **3 horizontal moves**: Since there are 3 horizontal moves, there would be 7 vertical moves. There are four possible endpoints, (-3, 7), (-3, -7), (3, 7), and (3, -7).

- **4 horizontal moves**: Since there are 4 horizontal moves, there would be 6 vertical moves. There are four possible endpoints, (-4, 6), (-4, -6), (4, 6), and (4, -6).

- **5 horizontal moves**: Since there are 5 horizontal moves, there would be 5 vertical moves. There are four possible endpoints, (-5, 5), (-5, -5), (5, 5), and (5, -5).

- **6 horizontal moves**: Since there are 6 horizontal moves, there would be 4 vertical moves. There are four possible endpoints, (-6, 4), (-6, -4), (6, 4), and (6, -4).

- **7 horizontal moves**: Since there are 7 horizontal moves, there would be 3 vertical moves. There are four possible endpoints, (-7, 3), (-7, -3), (7, 3), and (7, -3).

- **8 horizontal moves**: Since there are 8 horizontal moves, there would be 2 vertical moves. There are four possible endpoints, (-8, 2), (-8, -2), (8, 2), and (8, -2).

- **9 horizontal moves**: Since there are 9 horizontal moves, there would be 1 vertical move. There are four possible endpoints, (-9, 1), (-9, -1), (9, 1), and (9, -1).

- **10 horizontal moves**: Since there are 10 horizontal moves, there would be 0 vertical moves. There are two possible endpoints, (-10, 0), and (10, 0).

Therefore, there are a total of $2 + (4 \times 9) + 2 = 40$ points with integer coordinates that have a path length of 10 from $O$. 
Problem of the Week
Problem E
Apples and Bananas

In a survey, Grade 12 students were asked if they like apples. They were then asked if they like bananas. The information is summarized below.

- 30% of the students do not like apples
- 36 students do not like bananas.
- 60 students like both
- 48 students like one but not the other

How many students do not like apples and do not like bananas?
Problem
In a survey, Grade 12 students were asked if they like apples. They were then asked if they like bananas. The information is summarized: 30% of the students do not like apples; 36 students do not like bananas; 60 students like both; and 48 students like one but not the other. How many students do not like apples and do not like bananas?

Solution
We will set up a Venn Diagram.
Let \( z \) be the number of students that do not like apples and do not like bananas, \( x \) be the number of students that like only apples and \( y \) be the number of students that like only bananas. The 60 is the number of students who like both apples and bananas.
The number of students that do not like bananas is equal to the number of students outside the circle that is labelled bananas. This is \( x + z \). Therefore \( x + z = 36 \). Call this equation (1).
The number of students who like one but not the other is \( x + y = 48 \). Call this equation (2).
The total number of students is \( x + y + z + 60 \) and 30% of this is \( 0.3(x + y + z + 60) \). This is equal to the number of students that do not like apples, \( y + z \). Therefore, \( 0.3(x + y + z + 60) = y + z \). Call this equation (3).
Subtracting (1) from (2), we get \( y - z = 12 \). Therefore, \( y = z + 12 \). Call this equation (4).
Substituting (1) into (3) we get \( 0.3(36 + y + 60) = y + z \), or \( 0.3(96 + y) = y + z \).
Now substitute equation (4) for \( y \),
\[
0.3(96 + (z + 12)) = (z + 12) + z \\
28.8 + 0.3z + 3.6 = 2z + 12 \\
20.4 = 1.7z \\
z = 12
\]
Therefore, there are 12 students who do not like apples and do not like bananas.
Problem of the Week

Problem E

Triple Number Sums

The set \{3, 6, 9, 12, 15, \ldots , 2016, 2019\} contains all of the multiples of three from 3 to 2019.

Three distinct numbers are chosen from the set to form a sum. How many different sums can be formed?
Problem of the Week
Problem E and Solution
Triple Number Sums

Problem
The set \( \{3, 6, 9, 12, 15, \cdots, 2016, 2019\} \) contains all of the multiples of three from 3 to 2019. Three distinct numbers are chosen from the set to form a sum. How many different sums can be formed?

Solution
Since the set includes every positive multiple of three from 3 to 2019 and 2019 is the largest number, then there are \( 2019 \div 3 = 673 \) numbers in the set. Each number is of the form \( 3n \), for \( n = 1, 2, 3, \cdots, 673 \). The required sum is \( 3a + 3b + 3c \) where \( a, b, \) and \( c \) are three distinct numbers chosen from \( \{1, 2, 3, \cdots, 673\} \). But \( 3a + 3b + 3c = 3(a + b + c) \). We can reduce the problem to the much easier question of, “How many distinct integers can be formed by adding three numbers from \( \{1, 2, 3, \cdots, 673\}\)?”

Clearly, the smallest number is \( 1 + 2 + 3 = 6 \) and the largest number is \( 671 + 672 + 673 = 2016 \). It is reasonably easy to see that it is possible to get every number in between 6 and 2016 by:

a) increasing the sum by replacing a number with one that is 1 larger or,

b) decreasing the sum by replacing a number with one that is 1 smaller.

Therefore, all of the numbers from 6 to 2016 inclusive can be formed. The number of numbers that can be formed is 2011. (Some solvers may think that there are 2010 numbers. There are 2016 integers from 1 to 2016, inclusive. But this includes the five numbers 1 to 5. So there are \( 2016 - 5 = 2011 \) numbers from 6 to 2016.)

This answer, 2011, is the answer to the original problem as well. If \( a + b + c = 6 \) then \( 3(a + b + c) = 18 \). This is the smallest number that is the sum of the three smallest numbers, 3, 6 and 9, from the original set. If \( a + b + c = 2016 \) then \( 3(a + b + c) = 6048 \). This is the largest number that is the sum of the three largest numbers, 2013, 2016, and 2019, from the original set. Then every multiple of three from 18 to 6048 can be generated by adding three different numbers from the original set. (There are 2011 multiples of three from 18 to 6048, inclusive. And each of these can be obtained by adding three distinct numbers from the original set.)
Problem of the Week
Problem E
No Sevens

The number 254 does not contain the digit 7. The number 107 does contain the digit 7. Determine the sum of all integers between 1 and 2019, inclusive, that do not contain the digit 7.
Problem of the Week
Problem E and Solution
No Sevens

Problem
Determine the sum of all integers between 1 and 2019, inclusive, that do not contain the digit 7.

Solution
Solution 1
In this solution, we will use the fact that the sum of the integers from 1 to \( n \) is \( \frac{n(n+1)}{2} \).
Consider first the integers from 1 to 100. The sum of these integers is \( \frac{100(101)}{2} = 5050 \).
The integers from 1 to 100 which do contain the digit 7 are 7, 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 87, 97, whose sum is 1188. Therefore, the sum of the integers from 1 to 100 which do not contain the digit 7 is 5050 – 1188 = 3862. There are 81 integers from 101 to 200 not containing the digit 7 as well. Each of these is 100 more than a corresponding integer between 1 and 100 which does not contain the digit 7, so the sum of these 81 integers is 3862 + 81(100).
We can use this approach to determine the sum of the appropriate numbers in each range of 100, as shown in the table:

<table>
<thead>
<tr>
<th>Range</th>
<th>Number of Integers in Range that do not contain a 7</th>
<th>Sum of Integers in Range that do not contain a 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 to 100</td>
<td>81</td>
<td>3862</td>
</tr>
<tr>
<td>101 to 200</td>
<td>81</td>
<td>3862 + 81(100)</td>
</tr>
<tr>
<td>201 to 300</td>
<td>81</td>
<td>3862 + 81(200)</td>
</tr>
<tr>
<td>301 to 400</td>
<td>81</td>
<td>3862 + 81(300)</td>
</tr>
<tr>
<td>401 to 500</td>
<td>81</td>
<td>3862 + 81(400)</td>
</tr>
<tr>
<td>501 to 600</td>
<td>81</td>
<td>3862 + 81(500)</td>
</tr>
<tr>
<td>601 to 700</td>
<td>81</td>
<td>3862 + 81(600) – 700</td>
</tr>
<tr>
<td>701 to 800</td>
<td>1</td>
<td>800</td>
</tr>
<tr>
<td>801 to 900</td>
<td>81</td>
<td>3862 + 81(800)</td>
</tr>
<tr>
<td>901 to 1000</td>
<td>81</td>
<td>3862 + 81(900)</td>
</tr>
<tr>
<td>1001 to 1100</td>
<td>81</td>
<td>3862 + 81(1000)</td>
</tr>
<tr>
<td>1101 to 1200</td>
<td>81</td>
<td>3862 + 81(1100)</td>
</tr>
<tr>
<td>1201 to 1300</td>
<td>81</td>
<td>3862 + 81(1200)</td>
</tr>
<tr>
<td>1301 to 1400</td>
<td>81</td>
<td>3862 + 81(1300)</td>
</tr>
<tr>
<td>1401 to 1500</td>
<td>81</td>
<td>3862 + 81(1400)</td>
</tr>
<tr>
<td>1501 to 1600</td>
<td>81</td>
<td>3862 + 81(1500)</td>
</tr>
<tr>
<td>1601 to 1700</td>
<td>81</td>
<td>3862 + 81(1600) – 1700</td>
</tr>
<tr>
<td>1701 to 1800</td>
<td>1</td>
<td>1800</td>
</tr>
<tr>
<td>1801 to 1900</td>
<td>81</td>
<td>3862 + 81(1800)</td>
</tr>
<tr>
<td>1901 to 2000</td>
<td>81</td>
<td>3862 + 81(1900)</td>
</tr>
</tbody>
</table>
For 2001 to 2019, the only integers that contain the digit 7 are 2007 and 2017. Thus, the sum of the integers from 2001 to 2019 that do not contain a 7 is

\[
\]

Therefore, the overall sum of the integers from 1 to 2019 that do not contain the digit 7 is

\[
18(3862) + 81(16600) - 700 + 800 - 1700 + 1800 + 34166 = 1448482.
\]

**Solution 2**

Consider first the integers from 000 to 999 that do not contain the digit 7. (We can include 000 in this list as it will not affect the sum.)

Since each of the three digits has 9 possible values, there are \(9 \times 9 \times 9 = 729\) such integers.

If we fix any specific digit in any of the three positions, there will be exactly 81 integers with that digit in that position, as there are 9 possibilities for each of the remaining digits. (For example, there are 81 such integers ending in 0, 81 ending in 1, etc.)

We sum these integers by first summing the units digits, then summing the tens digits, and then summing the hundreds digits.

Since each of the 9 possible units digits occurs 81 times, the sum of the units digits column is

\[
81(0) + 81(1) + 81(2) + 81(3) + 81(4) + 81(5) + 81(6) + 81(8) + 81(9) = 81(38).
\]

Since each of the 9 possible tens digits occurs 81 times, the sum of the tens digits column is

\[
81(0 + 10 + 20 + 30 + 40 + 50 + 60 + 80 + 90) = 81(380).
\]

Similarly, the sum of the hundreds digits column is 81(3800).

Thus, the sum of the integers from 0 to 999 that do not contain the digit 7 is

\[
81(38) + 81(380) + 81(3800) = 81(38)(1 + 10 + 100) = 81(38)(111) = 341658.
\]

Each of the 729 integers from 1000 to 1999 which do not contain 7 is 1000 more than such an integer between 0 and 999. (There are again 729 of these integers as the first digit is fixed at 1, and each of the remaining three digits has 9 possible values.) Thus, the sum of these integers from 1000 to 1999 is equal to the sum of the corresponding ones from 0 to 999 plus 729(1000), or

\[
341658 + 729000 = 1070658.
\]

For 2000 to 2019, the only integers that contain the digit 7 are 2007 and 2017. Thus, the sum of the integers from 2000 to 2019 that do not contain the digit 7 is

\[
\]

Therefore, the sum of the integers from 1 to 2019 that do not include the digit 7 is

\[
341658 + 1070658 + 36166 = 1448482.
\]
Problem of the Week
Problem E
A Square Formation

The eight vertices of a regular octagon are randomly labelled $A, B, C, D, E, F, G,$ and $H$. Each letter is used exactly once. The figures below show three different ways to label the vertices of the octagon.

It can be shown that the shape created by drawing a line between every other vertex in a regular octagon is a square, and that this is the only way to use the vertices of a regular octagon to create a square within the octagon.

In the first labelling example, both $ACEG$ and $BDFH$ are squares, as illustrated below in the two diagrams on the left. These are the only two squares that can be created using that specific labelling of the octagon.

In the second example, both $ACEG$ and $BDFH$ are squares, as illustrated below in the middle two diagrams. These are the only two squares that can be created using that specific labelling of the octagon.

In the third example, both $ABCD$ and $EGHF$ are squares, as illustrated below in the two diagrams on the right. These are the only two squares that can be created using that specific labelling of the octagon.

If the vertices of a regular octagon are randomly labelled $A, B, C, D, E, F, G,$ and $H$ and each letter is used exactly once, what is the probability that $ABCD$ is a square?

That is, what is the probability that the figure formed by connecting the vertex labelled $A$ to the vertex labelled $B$, the vertex labelled $B$ to the vertex labelled $C$, the vertex labelled $C$ to the vertex labelled $D$, and the vertex labelled $D$ to the vertex labelled $A$, is a square?
Problem of the Week
Problem E and Solution
A Square Formation

Problem
The vertices of a regular octagon are randomly labelled $A, B, C, D, E, F, G,$ and $H$ and each letter is used exactly once. What is the probability that $ABCD$ is a square?

Solution
Solution 1

In order to determine the probability, we need to determine the number of ways to label the vertices of the regular octagon so that $ABCD$ forms a square and divide by the total number of ways the regular octagon can be labelled.

First, let’s determine the total number of ways that the vertices of a regular octagon can be labelled $A, B, C, D, E, F, G, H$, in some order.

Let’s start with the vertex on the top left. There are 8 possible ways to label it (it can be labelled as $A, B, C, D, E, F, G$ or $H$). Moving clockwise, the next vertex can be labelled 7 different ways (it can be assigned any letter other than the letter assigned to the previous vertex). Moving clockwise, the next vertex can be labelled 6 different ways (it can be assigned any letter other than the 2 that have already been used). Moving clockwise, the next vertex can be assigned 5 different letters, and so on. Once we reach the last vertex there will only be 1 letter left, so it can be assigned a letter only 1 way.

Therefore, there are $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40320$

different ways to label the regular octagon with the letters $A, B, C, D, E, F, G, H$ in some order.

Now, let’s determine how many of the 40320 labellings result in $ABCD$ forming a square.

Let’s suppose vertex $A$ is on the top left corner. Then there are two possible ways to label $B, C$ and $D$ so that $ABCD$ forms a square. They are shown below.
For each of these two cases, how many ways can the remaining 4 vertices be labelled? There are 4 choices for labelling the vertex to the right of A (it can be assigned E, F, G or H). Given the labelling of that vertex, moving clockwise, there are 3 choices for the next vertex, then 2 choices for the next and 1 choice for the last vertex.

Therefore, for each of the cases above, there are $4 \times 3 \times 2 \times 1 = 24$ ways to label the remaining vertices. Therefore, there are $24 + 24 = 48$ ways to label the regular octagon with A in the top left corner and $ABCD$ forming a square.

Using a similar argument, we can see that for any vertex that A can be assigned to, there will be 48 ways to label the regular octagon so that $ABCD$ forms a square. Since A can be assigned to 8 different vertices, there are $8 \times 48 = 384$ different ways to label the regular octagon so that $ABCD$ forms a square.

Therefore, the probability that $ABCD$ forms a square is $\frac{384}{40320} = \frac{1}{105}$.

Solution 2

The first solution counts the number of ways to create square $ABCD$ and divides by the total number of possible arrangements. This solution uses a more direct probability argument. Since the square labels A, B, C, and D are sequential, the A can go anywhere.

There is now a $\frac{2}{7}$ chance that the B will be placed in a location to create a square (either clockwise or counterclockwise around), given where the A is.

There is now a $\frac{1}{6}$ chance that the C will be placed in the only valid location, across from the A, and given that B is in an acceptable location.

There is now a $\frac{1}{5}$ chance that the D will be placed in the only valid location, given the A, B, and C are located appropriately.

The remaining assignments are irrelevant to square $ABCD$.

By multiplying the probabilities, we obtain the probability that $ABCD$ forms a square is

$$\frac{2}{7} \times \frac{1}{6} \times \frac{1}{5} = \frac{2}{210} = \frac{1}{105}.$$
Four cards each have a different positive integer printed on one side. Also, one card is purple, one card is blue, one card is green, and one card is yellow. (For those without a colour printer, the colour of the card is printed below the card.)

The four cards are placed on a table so that the numbered side is face down. Students look at the numbers on exactly three of the cards and then state the sum.

One student looked under the purple, blue and green cards and stated that the sum was 14. A second student looked under the blue, green and yellow cards and stated that the sum was 18. Finally, a third student looked under the purple, green and yellow cards and stated that the sum was 19.

Determine all possibilities for the positive integers on the purple, blue, green and yellow cards.

<table>
<thead>
<tr>
<th>PURPLE</th>
<th>BLUE</th>
<th>GREEN</th>
<th>YELLOW</th>
</tr>
</thead>
</table>

Problem

Four cards each have a different positive integer printed on one side. Also, one card is purple, one card is blue, one card is green, and one card is yellow. (For those without a colour printer, the colour of the card is printed below the card.) The four cards are placed on a table so that the numbered side is face down. Students look at the numbers on exactly three of the cards and then state the sum. One student looked under the purple, blue and green cards and stated that the sum was 14. A second student looked under the blue, green and yellow cards and stated that the sum was 18. Finally, a third student looked under the purple, green and yellow cards and stated that the sum was 19. Determine all possibilities for the positive integers on the purple, blue, green and yellow cards.

Solution

Solution 1

Let \(p\), \(b\), \(g\), and \(y\) represent the positive integers written on the purple, blue, green and yellow cards, respectively. It is given that \(p \neq b \neq g \neq y\).

It is also given that
\[
\begin{align*}
p + b + g &= 14 \quad (1) \\
b + g + y &= 18 \quad (2) \\
p + g + y &= 19 \quad (3)
\end{align*}
\]

(2) \(−\) (1) gives \(y - p = 4\) or equivalently, \(y = p + 4\) \quad (4)
(3) \(−\) (2) gives \(p - b = 1\) or equivalently, \(b = p - 1\) \quad (5)

Since \(p\), \(b\), \(g\) and \(y\) are all different positive integers, let’s look at the possible values of \(p\) and calculate the values of \(b, g\) and \(y\) in each case:

<table>
<thead>
<tr>
<th>Purple Card</th>
<th>Blue Card</th>
<th>Green Card</th>
<th>Yellow Card</th>
<th>Possible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(b = p - 1) \quad (5)</td>
<td>(g = 14 - p - b) (from (1))</td>
<td>(y = p + 4) \quad (4)</td>
<td>Yes or No</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>13</td>
<td>5</td>
<td>No ((b \neq 0))</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>11</td>
<td>6</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>Yes</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>No ((p = g))</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>Yes</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>1</td>
<td>11</td>
<td>Yes</td>
</tr>
</tbody>
</table>

We can stop here, because if \(p > 7\) then \(g < 0\), which is not valid.

Therefore, there are 5 different possibilities for the numbers on the purple, blue, green and yellow cards, respectively.

They are \((p, b, g, y) = (2, 1, 11, 6), (3, 2, 9, 7), (4, 3, 7, 8), (6, 5, 3, 10), (7, 6, 1, 11)\).
Solution 2

This solution is very similar to solution 1. The key difference is that, in this solution, we find an expression for each of \(b\), \(g\), and \(y\), in terms of \(p\). We then determine the smallest and largest possible values for \(p\). Based on these values we calculate the values of the other unknowns and determine the validity of each possibility.

Let \(p\), \(b\), \(g\), and \(y\) represent the positive integers written on the purple, blue, green and yellow cards, respectively. It is given that \(p \neq b \neq g \neq y\).

It is also given that
\[
\begin{align*}
p + b + g &= 14 \quad (1) \\
b + g + y &= 18 \quad (2) \\
p + g + y &= 19 \quad (3)
\end{align*}
\]

(2) \(-\) (1) gives \(y - p = 4\) and \(y = p + 4\) follows. (4)
(3) \(-\) (2) gives \(p - b = 1\) and \(b = p - 1\) follows. (5)

Substitute for \(y\) from (4) and \(b\) from (5) into (2) to get \(g\) in terms of \(p\).
\[
\begin{align*}
b + g + y &= 18 \\
(p - 1) + g + (p + 4) &= 18 \\
2p + g + 3 &= 18 \\
g &= 15 - 2p
\end{align*}
\]

Since \(b = p - 1\) and \(b\) is a positive integer, the smallest positive integer value for \(p\) will be 2. Otherwise, \(b \leq 0\).

Since \(g = 15 - 2p\) and \(g\) is a positive integer, the largest positive integer value for \(p\) will be 7. Otherwise, \(g \leq 0\).

Therefore, the only possible values for \(p\) are \(\{2, 3, 4, 5, 6, 7\}\).

We will now look at each possible value of \(p\) and calculate the values of \(b\), \(g\) and \(y\) in each case:

<table>
<thead>
<tr>
<th>Purple Card</th>
<th>Blue Card</th>
<th>Green Card</th>
<th>Yellow Card</th>
<th>Possible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(b = p - 1)</td>
<td>(g = 15 - 2p)</td>
<td>(y = p + 4)</td>
<td>Yes or No</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>11</td>
<td>6</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>No ((p = g))</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>1</td>
<td>11</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
</tbody>
</table>

Therefore, there are 5 different possibilities for the numbers on the purple, blue, green and yellow cards, respectively.

They are \((p, b, g, y) = (2, 1, 11, 6), (3, 2, 9, 7), (4, 3, 7, 8), (6, 5, 3, 10), (7, 6, 1, 11)\).
Problem of the Week

Problem E

Group of Six

$ABC$ is a three-digit integer whose first digit is $A$, second digit is $B$, and third digit is $C$. Similarly, $DEF$ is a three-digit integer whose first digit is $D$, second digit is $E$, and third digit is $F$.

We are given that

\[
\begin{array}{c}
A B C \\
+ D E F \\
\hline
1 2 3 4
\end{array}
\]

How many 6-tuples $(A, B, C, D, E, F)$ are there with $A > D$, $B > E$, and $C > F$ that make the above statement true?

$(8, 9, 3, 3, 4, 1)$
Problem of the Week
Problem E and Solution
Group of Six

Problem

$ABC$ is a three-digit integer whose first digit is $A$, second digit is $B$, and third digit is $C$. Similarly, $DEF$ is a three-digit integer whose first digit is $D$, second digit is $E$, and third digit is $F$. We are given that

$$
\begin{array}{c}
A B C \\
+ D E F \\
\hline
1 2 3 4
\end{array}
$$

How many 6-tuples $(A, B, C, D, E, F)$ are there with $A > D$, $B > E$, and $C > F$ that make the above statement true?

Solution

To solve this problem, we are going to look at each column starting with the units, then tens, and then finally the hundreds column.

Since $C + F$ ends in a 4, then $C + F = 4$ or $C + F = 14$. The value of $C + F$ cannot be 20 or more, as $C$ and $F$ are digits. In the case that $C + F = 14$, we “carry” a 1 to the tens column.

Since the result in the tens column is 3, then when there is no “carry” from the units column, $B + E$ ends in a 3, and when there is a “carry” from the units column $1 + B + E$ ends in a 3, so $B + E$ ends in a 2.

If $B + E$ ends in a 3, then $B + E = 3$ or $B + E = 13$. The value of $B + E$ cannot be 20 or more, as $B$ and $E$ are digits. In the case that $B + E = 13$, we “carry” a 1 to the hundreds column.

If $B + E$ ends in a 2, then $B + E = 2$ or $B + E = 12$. The value of $B + E$ cannot be 20 or more, as $B$ and $E$ are digits. In the case that $B + E = 12$, we “carry” a 1 to the hundreds column.

Since the result in the hundreds column is 12, then $A + D = 12$, or in the case when there was a “carry” from the tens column $1 + A + D = 12$, so $A + D = 11$.

We summarize this information in a tree.

```
Possible Sums

   C + F = 4
   /   \
  /     \
C + F = 14
  /   \
 B + E = 3  B + E = 13
     |         |         |
 A + D = 12  A + D = 11  A + D = 11
```

```
B + E = 13
  |         |
 A + D = 11  A + D = 11
```
We now look at the different possibilities for digits $A, B, C, D, E,$ and $F$ for each individual sum, with the restriction that $A > D$, $B > E$, and $C > F$.

<table>
<thead>
<tr>
<th>Sum</th>
<th>Possible solutions</th>
<th>Total Number of Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C + F = 4$</td>
<td>$C = 4, F = 0$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$C = 3, F = 1$</td>
<td></td>
</tr>
<tr>
<td>$C + F = 14$</td>
<td>$C = 9, F = 5$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$C = 8, F = 6$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 2$</td>
<td>$B = 2, E = 0$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$B = 3, E = 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$B = 2, E = 1$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 12$</td>
<td>$B = 9, E = 3$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$B = 8, E = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$B = 7, E = 5$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 13$</td>
<td>$B = 9, E = 4$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$B = 8, E = 5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$B = 7, E = 6$</td>
<td></td>
</tr>
<tr>
<td>$A + D = 11$</td>
<td>$A = 9, D = 2$</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$A = 8, D = 3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 7, D = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 6, D = 5$</td>
<td></td>
</tr>
<tr>
<td>$A + D = 12$</td>
<td>$A = 9, D = 3$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$A = 8, D = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 7, D = 5$</td>
<td></td>
</tr>
</tbody>
</table>

We can find the number of 6-tuples that are possible for the first branch of the tree. We have 2 choices for the units column. For each of these two choices, we have 2 choices for the tens column, so there are $2 \times 2 = 4$ possibilities. For each of these possibilities, we have 3 choices for the hundreds column. So the number of 6-tuples is $2 \times 2 \times 3 = 12$.

Similarly, for the second branch, we have 2 choices for the units column, 3 choices for the tens column, and 4 choices for the hundreds column. So the number of 6-tuples is $2 \times 3 \times 4 = 24$.

For the third branch, we have 2 choices for the units column, 1 choice for the tens column, and 3 choices for the hundreds column. So the number of 6-tuples is $2 \times 1 \times 3 = 6$.

For the fourth branch, we have 2 choices for the units column, 3 choices for the tens column, and 4 choices for the hundreds column. So the number of 6-tuples is $2 \times 3 \times 4 = 24$.

Therefore, the total number of 6-tuples is $12 + 24 + 6 + 24 = 66$. 
Problem of the Week
Problem E
Only Four?

A three-digit number, $abc$, is formed so that each of the digits in the number, $a$, $b$, and $c$, are different. Four of these three-digit numbers have the property that the product of the digits of the original number is the same as the two-digit number formed by the tens digit and units digit of the original number, namely $bc$.

One of the four numbers $abc$ is $236$, since $a \times b \times c = 2 \times 3 \times 6 = 36$, which is $bc$.

\[ 2 \times 3 \times 6 = 36 \]

Find the remaining three numbers.

It may be helpful to recall that any two-digit number of the form $bc$ can be represented by the sum $10b + c$. For example, $12 = 10(1) + 2$. 
Problem of the Week  
Problem E and Solution  
Only Four?

Problem  
A three-digit number, $abc$, is formed so that each of the digits in the number, $a$, $b$, and $c$, are different. Four of these three-digit numbers have the property that the product of the digits of the original number is the same as the two-digit number formed by the tens digit and units digit of the original number, namely $bc$.

One of the four numbers $abc$ is 236, since $a \times b \times c = 2 \times 3 \times 6 = 36$, which is $bc$.

$$2 \times 3 \times 6 = 36$$

Find the remaining three numbers.

Solution  
We know that there are only 4 answers, so we could attempt a trial and error approach to find the remaining 3 numbers. We will present a more systematic approach.

First, notice that since $a$, $b$ and $c$ are digits and must, therefore, be positive integers between 0 and 9. Also, since the product $a \times b \times c$ is a two-digit number, none of $a$, $b$ or $c$ can equal zero.

We are asked to find all three-digit numbers $abc$ such that $a \times b \times c = 10b + c$.

Since $b \neq 0$, we can divide by $b$ and the problem becomes equivalent to finding all integers $a, b, c$ with $1 \leq a, b, c \leq 9$ and $a, b$ and $c$ distinct such that

$$a \times c = 10 + \frac{c}{b}$$

Since $a$ and $c$ are integers, then so is $a \times c$ and we must have that $\frac{c}{b}$ is an integer as well. Therefore, for each possible value of $c$, we must have that $b$ divides exactly into $c$.

We will break the problem into cases based on the value of $c$, and then sub-cases based on what the value of $b$ can be for that particular $c$.

Case 1: $c = 1$

There are no values of $b$ where $\frac{c}{b}$ is an integer and $b \neq c$.

Case 2: $c = 2$

For $\frac{c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$. If $b = 1$ and $c = 2$,

$a \times b \times c = 10b + c$ becomes $2a = 12$ and so $a = 6$. Therefore, one of the three-digit numbers is 612.
Case 3: \( c = 3 \)
For \( \frac{c}{b} \) to be an integer and \( b \neq c \), we must have \( b = 1 \). If \( b = 1 \) and \( c = 3 \), 
\( a \times b \times c = 10b + c \) becomes \( 3a = 13 \) and so \( a = \frac{13}{3} \). Since \( a \) is not an integer, there is no solution in this case.

Case 4: \( c = 4 \)
For \( \frac{c}{b} \) to be an integer and \( b \neq c \), we must have \( b = 1 \) or \( b = 2 \).

Case i: \( b = 1 \)
In this case, \( a \times b \times c = 10b + c \) becomes \( 4a = 14 \) and so \( a = \frac{7}{2} \). Since \( a \) is not an integer, there is no solution in this case.

Case ii: \( b = 2 \)
In this case, \( a \times b \times c = 10b + c \) becomes \( 8a = 24 \) and so \( a = 3 \). Therefore, one of the three-digit numbers is 324.

Case 5: \( c = 5 \)
For \( \frac{c}{b} \) to be an integer and \( b \neq c \), we must have \( b = 1 \). If \( b = 1 \) and \( c = 5 \), 
\( a \times b \times c = 10b + c \) becomes \( 5a = 15 \) and so \( a = 3 \). Therefore, one of the three-digit numbers is 315.

Case 6: \( c = 6 \)
For \( \frac{c}{b} \) to be an integer and \( b \neq c \), we must have \( b = 1 \), \( b = 2 \) or \( b = 3 \).

Case i: \( b = 1 \)
In this case, \( a \times b \times c = 10b + c \) becomes \( 6a = 16 \) and so \( a = \frac{8}{3} \). Since \( a \) is not an integer, there is no solution in this case.

Case ii: \( b = 2 \)
In this case, \( a \times b \times c = 10b + c \) becomes \( 12a = 26 \) and so \( a = \frac{13}{6} \). Since \( a \) is not an integer, there is no solution in this case.

Case iii: \( b = 3 \)
In this case, \( a \times b \times c = 10b + c \) becomes \( 18a = 36 \) and so \( a = 2 \). Therefore, one of the three-digit numbers is 236. This is the number given in the example.

We can actually stop here since we have found 4 different three-digit numbers that satisfy the conditions outlined in the problem. If we had not been given the number of possible solutions, we would need to continue by checking cases when \( c = 7 \), \( c = 8 \) and \( c = 9 \).

Therefore, the 4 three-digit numbers that satisfy the conditions of the problem are 612, 324, 315, and 236.
Functions
(includes Trigonometry)
Problem of the Week

Problem E

An Uphill Struggle

The following information is known about \( \triangle OBC \):

- \( O \) is at the origin, and points \( B \) and \( C \) lie in the first quadrant;
- \( \triangle OBC \) is an isosceles right triangle with \( OB = BC \) and \( \angle OBC = 90^\circ \); and
- the hypotenuse \( OC \) is on a line segment with slope 3.

Determine the slope of line segment \( OB \).
Problem of the Week
Problem E and Solution
An Uphill Struggle

Problem
The following information is known about $\triangle OBC$: $O$ is at the origin and points $B$ and $C$ lie in the first quadrant; $\triangle OBC$ is an isosceles right triangle with $OB = BC$ and $\angle OBC = 90^\circ$; and the hypotenuse $OC$ is on a line segment with slope 3. Determine the slope of line segment $OB$.

Solution
We present three solutions. The first involves a construction. The second solution follows after making an assumption. The third solution uses trigonometry. The formula used in the third solution may not be familiar to all students.

Solution 1
Draw a line through $C$ parallel to the $x$-axis, intersecting the $y$-axis at $R$.
Draw a line through $B$ parallel to the $y$-axis, intersecting the $x$-axis at $P$ and intersecting the first line through $R$ and $C$ at $Q$.
This construction creates rectangle $OPQR$.
In $\triangle CQB$, let $\angle QCB = \alpha$ and $\angle QBC = \beta$. Since $OPQR$ is a rectangle, $\angle BQC = 90^\circ$ and $\triangle CQB$ is a right angled triangle. It follows that $\alpha + \beta = 90^\circ$.
$\angle QBP$ is a straight angle so $\angle QBC + \angle CBO + \angle OBP = 180^\circ$.
Substituting, we obtain $\beta + 90^\circ + \angle OBP = 180^\circ$ which simplifies to $\beta + \angle OBP = 90^\circ$.
But $\alpha + \beta = 90^\circ$ so it follows that $\angle OBP = \alpha$. Then in right triangle $BPO$, we get $\angle BOP = \beta$.
In $\triangle CQB$ and $\triangle BPO$, since $\angle QCB = \angle OBP = \alpha$, $\angle QBC = \angle BOP = \beta$, and $BC = OB$ (given), then $\triangle CQB \cong \triangle BPO$.
From the triangle congruence, we get $CQ = BP = b$ and $QB = OP = a$.
In rectangle $OPQR$, $RC + CQ = OP$. Substituting, we obtain $RC + b = a$ and $RC = a - b$ follows.
All of this information is shown on the diagram above.
The coordinates of $C$ are $(a - b, a + b)$ and the coordinates of $B$ are $(a, b)$.
We know the slope of $OC = 3$, so $\frac{a + b}{a - b} = 3$. Simplifying, we obtain $a + b = 3a - 3b$ and $a = 2b$ follows.
Then the slope of $OB = \frac{b}{a} = \frac{b}{2b} = \frac{1}{2}$. 
Solution 2

Since $OC$ is a line segment with slope 3, with $O$ at the origin and $C$ in the first quadrant, the coordinates of $C$ will be of the form $(a, 3a)$, where $a$ is some positive number. We will do our calculations with $a = 2$. Then the length of $OC$ is $2\sqrt{10}$. Let $B$ be the point $(p, q)$.

Let $M$ be the midpoint of $OC$. Then $M$ is the point $(1, 3)$. It follows that $OM = MC = \frac{1}{2}OC = \sqrt{10}$.

In an isosceles right triangle, the line segment joining the midpoint of the hypotenuse to the opposite vertex is perpendicular to the hypotenuse and has length equal to half the length of the hypotenuse. (If this result is not known, it is easily shown using congruent triangles.)

It follows that $MB \perp OC$ and $MB = \sqrt{10}$.

Since $MB \perp OC$ and the slope of $OC$ is 3, then the slope of $MB$ is $-\frac{1}{3}$. We can find the equation of the line containing $M(1, 3)$ with slope $-\frac{1}{3}$ by substituting into $y = mx + b$.

$$3 = -\frac{1}{3}(1) + b$$
$$9 = -1 + 3b$$
$$10 = 3b$$
$$\frac{10}{3} = b$$

The equation of the line containing $MB$ is $y = -\frac{1}{3}x + \frac{10}{3}$.

Since $B(p, q)$ is on this line, $q = -\frac{1}{3}p + \frac{10}{3}$. (1)

The length of $MB$ is $\sqrt{10}$. Using $M(1, 3)$ and $B(p, -\frac{1}{3}p + \frac{10}{3})$,

$$MB^2 = (p - 1)^2 + \left(-\frac{1}{3}p + \frac{10}{3} - 3\right)^2$$
$$10 = (p - 1)^2 + \left(-\frac{1}{3}p + \frac{1}{3}\right)^2$$
$$10 = (p - 1)^2 + \frac{1}{9}(p - 1)^2$$
$$10 = \frac{10}{9}(p - 1)^2$$
$$9 = (p - 1)^2$$
$$\pm 3 = p - 1$$

It follows that $p = 4$ or $p = -2$. Since $B$ is in quadrant 1, $p = -2$ is inadmissible. Therefore, $p = 4$. Substituting in (1), $q = 2$ and $B$ is the point $(4, 2)$. Thus, the slope of $OB = \frac{2}{4} = \frac{1}{2}$. 

Solution 3

Since $\triangle OBC$ is an isosceles right triangle with $\angle OBC = 90^\circ$, then $\angle BOC = \angle BCO = 45^\circ$.

Let $\theta$ represent the angle that $OC$ makes with the positive $x$-axis. Since the slope of $OC = 3$, then $\tan \theta = 3$, since the tangent of an angle is equal to the slope of a line that makes this angle with the horizontal (the positive $x$-axis in this case).

The angle that $OB$ makes with the positive $x$-axis is $\theta - 45^\circ$. The slope of $OB$ will equal $\tan(\theta - 45^\circ)$.

Using the fact that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$,

\[
\tan(\theta - 45^\circ) = \frac{\tan \theta - \tan 45^\circ}{1 + \tan \theta \tan 45^\circ}
\]

\[
= \frac{3 - 1}{1 + 3(1)}
\]

\[
= \frac{2}{4}
\]

\[
= \frac{1}{2}
\]

Therefore, the slope of $OB = \tan(\theta - 45^\circ) = \frac{1}{2}$.
An aircraft carrier is traveling at an average speed of 34 km/h on a course that is 60° west of south. A helicopter leaves the carrier and travels due north at an average speed of 200 km/h. The carrier maintains its course and average speed. The helicopter has enough fuel for five hours of flying and it maintains its constant speed of 200 km/h.

What is the maximum distance north the helicopter can travel, so that the fuel remaining will allow a safe return to the aircraft carrier?
Problem of the Week
Problem E and Solution
Return Safely

Problem
An aircraft carrier is traveling at an average speed of 34 km/h on a course that is $60^\circ$ west of south. A helicopter leaves the carrier and travels due north at an average speed of 200 km/h. The carrier maintains its course and average speed. The helicopter has enough fuel for five hours of flying and it maintains its constant speed of 200 km/h.

What is the maximum distance north the helicopter can travel, so that the fuel remaining will allow a safe return to the aircraft carrier?

Solution
Solution 1
We know that the total time available is 5 hours and in that time the aircraft carrier will travel $5 \times 34 = 170$ km.

Let $t$ represent the time, in hours, that the helicopter travels due north. Then, the helicopter will have $(5 - t)$ hours to return to the carrier. The helicopter will fly $200t$ km due north and $200(5 - t) = (1000 - 200t)$ km in a direction which allows the helicopter to return to the carrier.

On the diagram, $O$ represents the point at which the helicopter leaves the carrier, $A$ represents the farthest point north that the helicopter reaches and $B$ represents the point that the aircraft carrier reaches in 5 hours. $\angle AOB = 180^\circ - 60^\circ = 120^\circ$. So $OA = 200t$ km, $OB = 170$ km, and $AB = (1000 - 200t)$ km. If we let $d = 200t$, we can simplify $OA$ to $d$ and $AB$ to $(1000 - d)$.

Using the cosine law,

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos(\angle AOB)$$

$$= d^2 + 170^2 - 2(170)(d)\cos(120^\circ)$$

$$= d^2 + 28900 - 340d \left(\frac{-1}{2}\right)$$

$$= d^2 + 28900 - 170d$$

$$= 1000000 - 2000d + 170^2$$

$$= 1000000 - 2000d + 28900$$

$$= 971100$$

$$d = \frac{971100}{2170}$$

$$d \approx 447.51 \text{ km}$$

The helicopter can travel about 447 km north before it must return to the final location of the carrier. Since $d = 200t$, we have $t = d \div 200 \approx 2 \text{ h} 14 \text{ min}$. (Both the distance and the time have been rounded down to allow for a safe return.) In terms of time, the pilot can fly north for about 2 hours and 14 minutes before having to return to the aircraft carrier.
Solution 2

Let \( t \) represent the maximum time that the helicopter can travel north. Let \( O \) represent the point where the helicopter leaves the carrier. Let \( A \) represent the most northerly point to which the helicopter could fly to before it must return to the carrier. Let \( B \) represent the location of the carrier after it travels five hours.

As in solution 1, we determine that the carrier travels \( 5 \times 34 = 170 \) km in five hours. We will describe the positions of the helicopter and the aircraft carrier in terms of coordinates. We know that the carrier is traveling through the third quadrant at an angle of 60° from the negative \( y \)-axis and 30° from the negative \( x \)-axis. The standard position angle related to these two angles would be \( 180° + 30° = 210° \). We can determine the coordinates of \( B \), the location of the carrier after traveling 5 hours, using

\[
x = r \cos \theta = 170 \cos 210° = 170 \left( -\frac{\sqrt{3}}{2} \right) = -85\sqrt{3}
\]

\[
y = r \sin \theta = 170 \sin 210° = 170 \left( -\frac{1}{2} \right) = -85
\]

The coordinates of \( B \) are \((-85\sqrt{3}, -85)\).

Since the helicopter is traveling due north at 200 km/h for \( t \) hours, it travels \( 200t \) km and the coordinates of \( A \) are \((0, 200t)\). We can now calculate the distance from \( A(0, 200t) \) to \( B(-85\sqrt{3}, -85) \).

\[
(AB)^2 = \left( 0 - (-85\sqrt{3}) \right)^2 + (200t - (-85))^2
= (85\sqrt{3})^2 + (200t + 85)^2
= 7225(3) + 40000t^2 + 34000t + 7225
= 21675 + 40000t^2 + 34000t + 7225
= 40000t^2 + 34000t + 28900
= 100(400t^2 + 340t + 289)
AB = 10\sqrt{400t^2 + 340t + 289}, \ AB > 0
\]

The time from \( A \) to \( B \) is \( \frac{10\sqrt{400t^2 + 340t + 289}}{200} = \sqrt{400t^2 + 340t + 289} \). We already know that the time from \( O \) to \( A \) is \( t \) and that the total time is 5 hours. Then,

\[
t + \frac{\sqrt{400t^2 + 340t + 289}}{20} = 5
\]

\[
20t + \sqrt{400t^2 + 340t + 289} = 100
\]

\[
\sqrt{400t^2 + 340t + 289} = 100 - 20t
\]

Squaring both sides,

\[
400t^2 + 340t + 289 = 10000 - 4000t + 400t^2
\]

\[
4340t = 9711
\]

\[
t = \frac{9711}{4340}
\]

The maximum distance the helicopter can travel north is \( 200t = \frac{971100}{2170} \) ≈ 447.51 km and the helicopter should not go north for more than 2 hours and 14 minutes.
Problem of the Week
Problem E
Parabolic Move

The parabola $y = -x^2 + 4$ has vertex labelled $P$ and intersects the $x$-axis at the points labelled $A$ and $B$. The parabola is translated from its original position so that its vertex moves along the line $y = x + 4$ to the point $Q$. In this new position, the parabola intersects the $x$-axis at the points labelled $B$ and $C$. Determine the coordinates of $C$. 
Problem of the Week
Problem E and Solution
Parabolic Move

Problem
The parabola \( y = -x^2 + 4 \) has vertex labelled \( P \) and intersects the \( x \)-axis at the points labelled \( A \) and \( B \). The parabola is translated from its original position so that its vertex moves along the line \( y = x + 4 \) to the point \( Q \). In this new position, the parabola intersects the \( x \)-axis at the points labelled \( B \) and \( C \). Determine the coordinates of \( C \).

Solution
For the original parabola \( y = -x^2 + 4 \), the vertex is \( P(0, 4) \) and the \( x \)-intercepts are \( A(-2, 0) \) and \( B(2, 0) \).
Let the vertex of the translated parabola be \( Q(q, p) \). Since the new parabola is a translation of the original, the equation of this new parabola is \( y = -(x-q)^2 + p \).
Since \( Q \) lies on the line \( y = x + 4 \), we have \( p = q + 4 \) and the equation of the new parabola is \( y = -(x-q)^2 + q + 4 \).
Since \( B(2,0) \) lies on the new parabola, we can substitute \((2,0)\) into this equation:

\[
\begin{align*}
0 &= -(2-q)^2 + q + 4 \\
0 &= -(q^2 - 4q + 4) + q + 4 \\
0 &= -q^2 + 4q - 4 + q + 4 \\
0 &= -q^2 + 5q \\
0 &= -q(q-5)
\end{align*}
\]

Therefore, \( q = 0 \) or \( q = 5 \). The value \( q = 0 \) corresponds to point \( P(0,4) \) in the original parabola. Therefore, \( q = 5 \) and the vertex of the new parabola is \((5,9)\) and the equation of this parabola is \( y = -(x-5)^2 + 9 \).
Since \( C \) is an \( x \)-intercept of this parabola, to determine \( C \) set \( y = 0 \) in the equation for the parabola and solve for \( x \):

\[
\begin{align*}
0 &= -(x-5)^2 + 9 \\
(x-5)^2 &= 9 \\
x - 5 &= \pm 3 \\
x &= 8, 2
\end{align*}
\]

The value \( x = 2 \) corresponds to point \( B \), and the value \( x = 8 \) corresponds to point \( C \). Therefore, the coordinates of \( C \) are \((8,0)\).
Problem of the Week
Problem E
Up, Down, Up, Down, ...

The sequence \( \{1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \cdots \} \) is called an alternating sequence since the terms alternate positive and negative. In this particular sequence, the value of any term in an odd position in the sequence is the term number and the value of any term in an even position in the sequence is the term number multiplied by \(-1\).

Let \( S(n) \) be the sum of the first \( n \) terms of the sequence. Then it follows that

\[
\begin{align*}
S(1) &= 1 \\
S(3) &= 1 - 2 + 3 = 2 \\
S(5) &= 1 - 2 + 3 - 4 + 5 = 3 \\
S(7) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 = 4 \\
S(9) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 = 5 \\
S(2) &= 1 - 2 = -1 \\
S(4) &= 1 - 2 + 3 - 4 = -2 \\
S(6) &= 1 - 2 + 3 - 4 + 5 - 6 = -3 \\
S(8) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 = -4 \\
S(10) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 = -5
\end{align*}
\]

The following are two unproven results from the above sums.

If \( n \) is an even positive integer, i.e., \( n = 2k \) where \( k \) is a positive integer, then
\( S(n) = S(2k) = -k \). For example, \( S(6) = S(2 \times 3) = -3 \).

If \( n \) is an odd positive integer, i.e., \( n = 2m - 1 \) where \( m \) is a positive integer, then
\( S(n) = S(2m - 1) = m \). For example, \( S(9) = S(2 \times 5 - 1) = 5 \).

Suppose that \( a \) and \( b \) are any positive integers. Using the two results above, show that \( S(a) + S(b) + S(a + b) = 1 \) when both \( a \) and \( b \) are odd.

As an extension, show that \( S(a) + S(b) + S(a + b) = 1 \) only when both \( a \) and \( b \) are odd. As a further extension, prove that the two given results are true.
Problem

The sequence \{1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots\} is called an alternating sequence since the terms alternate positive and negative. In this particular sequence, the value of any term in an odd position in the sequence is the term number and the value of any term in an even position in the sequence is the term number multiplied by -1.

Let \( S(n) \) be the sum of the first \( n \) terms of the sequence. Then it follows that

\[
\begin{align*}
S(1) &= 1 \\
S(3) &= 1 - 2 + 3 = 2 \\
S(5) &= 1 - 2 + 3 - 4 + 5 = 3 \\
S(7) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 = 4 \\
S(9) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 = 5 \\
S(2) &= 1 - 2 = -1 \\
S(4) &= 1 - 2 + 3 - 4 = -2 \\
S(6) &= 1 - 2 + 3 - 4 + 5 - 6 = -3 \\
S(8) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 = -4 \\
S(10) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 = -5
\end{align*}
\]

Show that \( S(a) + S(b) + S(a + b) = 1 \) when both \( a \) and \( b \) are odd positive integers.

Solution

Suppose \( a \) and \( b \) are both odd positive integers. Then \( a = 2k - 1 \) and \( b = 2m - 1 \), for some positive integers \( k \) and \( m \). If \( k = m \), then \( a = b \), but we want to prove the result true for any odd positive integers \( a \) and \( b \), so it may be the case that \( k \neq m \).

We have \( S(a) = S(2k - 1) = k \) and \( S(b) = S(2m - 1) = m \).

Also, \( a + b = (2k - 1) + (2m - 1) = 2k + 2m - 2 = 2(k + m - 1) \) is even, so

\( S(a + b) = S(2(k + m - 1)) = -(k + m - 1) = -k - m + 1 \).

Then, \( S(a) + S(b) + S(a + b) = k + m + (-k - m + 1) = 1 \).

Therefore, \( S(a) + S(b) + S(a + b) = 1 \) whenever \( a \) and \( b \) are odd positive integers.

Extension

Since we have formulas for \( S(a) \) and \( S(b) \) based on whether \( a \) and \( b \) are even or odd, we will break into cases based on the parity of \( a \) and \( b \) to show that none of the other possible cases make \( S(a) + S(b) + S(a + b) = 1 \) true.

Case 1: \( a \) and \( b \) are both even positive integers

Let \( a = 2k \) and \( b = 2m \), for some positive integers \( k \) and \( m \).

We have \( S(a) = S(2k) = -k \) and \( S(b) = S(2m) = -m \).

Also, \( a + b = 2k + 2m = 2(k + m) \) is even, so \( S(a + b) = S(2(k + m)) = -(k + m) = -k - m \).

Then \( S(a) + S(b) + S(a + b) = -k + m + (-k - m) = -2k - 2m = -2(k + m) \). Since \( k \) and \( m \) are positive integers, \( k + m \) is also a positive integer, so \( -2(k + m) \) is a negative integer and so we cannot have \( -2(k + m) = 1 \). Therefore, \( S(a) + S(b) + S(a + b) \neq 1 \).
Case 2: \(a\) is even and \(b\) is odd

Let \(a = 2k\) and \(b = 2m - 1\), for some positive integers \(k\) and \(m\).
We have \(S(a) = S(2k) = -k\) and \(S(b) = S(2m - 1) = m\).
Also, \(a + b = 2k + (2m - 1) = 2(k + m) - 1\) is odd, so \(S(a + b) = S(2(k + m) - 1) = k + m\).
Then \(S(a) + S(b) + S(a + b) = -k + m + (k + m) = 2m\). Since \(m\) is an integer, we cannot have \(2m = 1\) (then \(m = \frac{1}{2}\)). Therefore, \(S(a) + S(b) + S(a + b) \neq 1\).

Case 3: \(a\) is odd and \(b\) is even

Let \(a = 2k - 1\) and \(b = 2m\), for some positive integers \(k\) and \(m\).
We have \(S(a) = S(2k - 1) = k\) and \(S(b) = S(2m) = -m\).
Also, \(a + b = (2k - 1) + 2m = 2(k + m) - 1\) is odd, so \(S(a + b) = S(2(k + m) - 1) = k + m\).
Then \(S(a) + S(b) + S(a + b) = k - m + (k + m) = 2k\). Since \(k\) is an integer, we cannot have \(2k = 1\) (then \(k = \frac{1}{2}\)). Therefore, \(S(a) + S(b) + S(a + b) \neq 1\).

Therefore, \(S(a) + S(b) + S(a + b) = 1\) is only true when both \(a\) and \(b\) are odd positive integers.

We will now prove the first fact given in the question.

We will show that for numbers of the form \(n = 2k\), we have \(S(n) = S(2k) = -k\).

Numbers of the form \(n = 2k\):

\[
S(2k) = 1 - 2 + 3 - 4 + 5 + \ldots + (2k - 1) - 2k \\
= (1 + 3 + 5 + \ldots + (2k - 1)) - (2 + 4 + 6 + \ldots + 2k) \quad \text{grouping even and odd numbers} \\
= (1 + 3 + 5 + \ldots + (2k - 1)) - 2(1 + 2 + 3 + \ldots + k) \quad \text{factor out a 2 from the even numbers} \\
= (1 + 3 + 5 + \ldots + (2k - 1)) + (2 + 4 + 6 + \ldots + 2k) - (2 + 4 + 6 + \ldots + 2k) - 2(1 + 2 + 3 + \ldots + k) \quad \text{add and subtract the even numbers from 2 to 2k} \\
= (1 + 2 + 3 + 4 + 5 + \ldots + (2k - 1) + 2k) - (2 + 4 + 6 + \ldots + 2k) - 2(1 + 2 + 3 + \ldots + k) \\
= (1 + 2 + 3 + 4 + 5 + \ldots + (2k - 1) + 2k) - 2(1 + 2 + 3 + \ldots + k) \\
= (1 + 2 + 3 + 4 + 5 + \ldots + (2k - 1) + 2k) - 4(1 + 2 + 3 + \ldots + k)
\]

Now, using the formula for the sum of the first \(n\) integers, that is

\[
1 + 2 + 3 + \ldots + (n - 1) + n = \frac{n(n + 1)}{2}
\]

we have

\[
S(2k) = (1 + 2 + 3 + 4 + 5 + \ldots + (2k - 1) + 2k) - 4(1 + 2 + 3 + \ldots + k) \\
= \frac{(2k)(2k + 1)}{2} - 4 \left( \frac{k(k + 1)}{2} \right) \\
= k(2k + 1) - 2k(k + 1) \\
= 2k^2 + k - 2k^2 - 2k \\
= -k
\]

The proof that \(S(n) = S(2k - 1) = k\) for numbers of the form \(n = 2k - 1\) is similar.
We’ll leave that for you to try on your own.
Problem of the Week
Problem E
Cornered

A cube of side length 8 cm is shown with a corner removed by making a cut through the midpoints of three adjacent sides.

Similar cuts are made at each of the remaining corners of the cube.

Determine the increase or decrease in total surface area as a result of slicing the eight corners off the original cube.
Problem of the Week
Problem E and Solution
Cornered

Problem
A cube of side length 8 cm is shown above with a corner removed by making a cut through the midpoints of three adjacent sides. Similar cuts are made at each of the remaining corners of the cube.

Determine the increase or decrease in total surface area as a result of slicing the eight corners off the original cube.

Solution
To determine the increase or decrease in surface area, we need to only look at one corner, find the surface area increase or decrease there and then multiply the result by eight to account for the eight corners. At each corner, the surface area of three right-angled triangles (top image to the right) is removed and replaced by the surface area of a single triangle (middle image to the right).

Since the cut is made from the midpoints of three adjacent sides, each right triangle has base of length 4 cm and a height of length 4 cm. The resulting area of one of the right triangles is \( \frac{1}{2}(4)(4) = 8 \text{ cm}^2 \).

The hypotenuse of the right triangle is found using Pythagoras’ Theorem, \( \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2} \text{ cm} \). Since the length of the hypotenuse is the same in each of the three right triangles, the remaining triangle in the corner is equilateral with sides of length \( 4\sqrt{2} \text{ cm} \). The triangle looks like the bottom triangle shown to the right.

The altitude of an equilateral triangle right bisects the base. Let \( h \) be the height of the equilateral triangle. Using Pythagoras’ Theorem, \( h^2 = (4\sqrt{2})^2 - (2\sqrt{2})^2 = 32 - 8 = 24 \). Therefore \( h = \sqrt{24} = 2\sqrt{6} \text{ cm} \) (since \( h > 0 \)).

The area of the remaining triangle is \( \frac{1}{2}(4\sqrt{2})(2\sqrt{6}) = 4\sqrt{12} = 8\sqrt{3} \text{ cm}^2 \).

For each corner, the new surface area is increased by the area of the equilateral triangle and decreased by the areas of the three right triangles. Therefore, slicing off a corner changes the surface area by \( 8\sqrt{3} - 3 \times (8) = (8\sqrt{3} - 24) \text{ cm}^2 \). Since \( 8\sqrt{3} < 24 \), this result is negative and the surface area is decreased in each corner. Therefore, slicing off a corner decreases the surface area by \( (24 - 8\sqrt{3}) \text{ cm}^2 \).

Since there are eight corners, the total decrease in surface area is

\[ 8 \times (24 - 8\sqrt{3}) = 192 - 64\sqrt{3} = 64(3 - \sqrt{3}) \approx 81.1 \text{ cm}^2. \]

Therefore, as a result of cutting off each corner, the total surface area decreases by \( 64(3 - \sqrt{3}) \text{ cm}^2 \).
Geometry & Measurement
Problem of the Week
Problem E
An Uphill Struggle

The following information is known about \( \triangle OBC \):

- \( O \) is at the origin, and points \( B \) and \( C \) lie in the first quadrant;
- \( \triangle OBC \) is an isosceles right triangle with \( OB = BC \) and \( \angle OBC = 90^\circ \); and
- the hypotenuse \( OC \) is on a line segment with slope 3.

Determine the slope of line segment \( OB \).
Problem of the Week
Problem E and Solution
An Uphill Struggle

Problem
The following information is known about \( \triangle OBC \): \( O \) is at the origin and points \( B \) and \( C \) lie in the first quadrant; \( \triangle OBC \) is an isosceles right triangle with \( OB = BC \) and \( \angle OBC = 90^\circ \); and the hypotenuse \( OC \) is on a line segment with slope 3. Determine the slope of line segment \( OB \).

Solution
We present three solutions. The first involves a construction. The second solution follows after making an assumption. The third solution uses trigonometry. The formula used in the third solution may not be familiar to all students.

Solution 1
Draw a line through \( C \) parallel to the \( x \)-axis, intersecting the \( y \)-axis at \( R \).
Draw a line through \( B \) parallel to the \( y \)-axis, intersecting the \( x \)-axis at \( P \) and intersecting the first line through \( R \) and \( C \) at \( Q \).
This construction creates rectangle \( OPQR \).
In \( \triangle CQB \), let \( \angle QCB = \alpha \) and \( \angle QBC = \beta \). Since \( OPQR \) is a rectangle, \( \angle BQC = 90^\circ \) and \( \triangle CQB \) is a right angled triangle. It follows that \( \alpha + \beta = 90^\circ \).
\( \angle QBP \) is a straight angle so \( \angle QBC + \angle CBO + \angle OBP = 180^\circ \).
Substituting, we obtain \( \beta + 90^\circ + \angle OBP = 180^\circ \) which simplifies to \( \beta + \angle OBP = 90^\circ \).
But \( \alpha + \beta = 90^\circ \) so it follows that \( \angle OBP = \alpha \). Then in right triangle \( BPO \), we get \( \angle BOP = \beta \).
In \( \triangle CQB \) and \( \triangle BPO \), since \( \angle QCB = \angle OBP = \alpha \), \( \angle QBC = \angle BOP = \beta \), and \( BC = OB \) (given), then \( \triangle CQB \cong \triangle BPO \).
From the triangle congruence, we get \( CQ = BP = b \) and \( QB = OP = a \).
In rectangle \( OPQR \), \( RC + CQ = OP \). Substituting, we obtain \( RC + b = a \) and \( RC = a - b \) follows.
All of this information is shown on the diagram above.
The coordinates of \( C \) are \( (a - b, a + b) \) and the coordinates of \( B \) are \( (a, b) \).
We know the slope of \( OC = 3 \), so \( \frac{a + b}{a - b} = 3 \). Simplifying, we obtain \( a + b = 3a - 3b \) and \( a = 2b \) follows.
Then the slope of \( OB = \frac{b}{a} = \frac{b}{2b} = \frac{1}{2} \).
Solution 2

Since $OC$ is a line segment with slope 3, with $O$ at the origin and $C$ in the first quadrant, the coordinates of $C$ will be of the form $(a,3a)$, where $a$ is some positive number. We will do our calculations with $a = 2$. Then the length of $OC$ is $2\sqrt{10}$. Let $B$ be the point $(p,q)$.

Let $M$ be the midpoint of $OC$. Then $M$ is the point $(1,3)$. It follows that $OM = MC = \frac{1}{2}OC = \sqrt{10}$.

In an isosceles right triangle, the line segment joining the midpoint of the hypotenuse to the opposite vertex is perpendicular to the hypotenuse and has length equal to half the length of the hypotenuse. (If this result is not known, it is easily shown using congruent triangles.)

It follows that $MB \perp OC$ and $MB = \sqrt{10}$.

Since $MB \perp OC$ and the slope of $OC$ is 3, then the slope of $MB$ is $-\frac{1}{3}$. We can find the equation of the line containing $M(1,3)$ with slope $-\frac{1}{3}$ by substituting into $y = mx + b$.

\[
\begin{align*}
3 &= \frac{1}{3}(1) + b \\
9 &= -1 + 3b \\
10 &= 3b \\
\frac{10}{3} &= b
\end{align*}
\]

The equation of the line containing $MB$ is $y = -\frac{1}{3}x + \frac{10}{3}$.

Since $B(p,q)$ is on this line, $q = -\frac{1}{3}p + \frac{10}{3}$. (1)

The length of $MB$ is $\sqrt{10}$. Using $M(1,3)$ and $B(p, -\frac{1}{3}p + \frac{10}{3})$,

\[
\begin{align*}
MB^2 &= (p - 1)^2 + \left(-\frac{1}{3}p + \frac{10}{3} - 3\right)^2 \\
(\sqrt{10})^2 &= (p - 1)^2 + \left(-\frac{1}{3}p + \frac{1}{3}\right)^2 \\
10 &= (p - 1)^2 + \left(-\frac{1}{3}(p - 1)\right)^2 \\
10 &= (p - 1)^2 + \frac{1}{9}(p - 1)^2 \\
10 &= \frac{10}{9}(p - 1)^2 \\
9 &= (p - 1)^2 \\
\pm 3 &= p - 1
\end{align*}
\]

It follows that $p = 4$ or $p = -2$. Since $B$ is in quadrant 1, $p = -2$ is inadmissible. Therefore, $p = 4$. Substituting in (1), $q = 2$ and $B$ is the point $(4,2)$. Thus, the slope of $OB = \frac{2}{4} = \frac{1}{2}$. 
Solution 3

Since $\triangle OBC$ is an isosceles right triangle with $\angle OBC = 90^\circ$, then $\angle BOC = \angle BCO = 45^\circ$.

Let $\theta$ represent the angle that $OC$ makes with the positive $x$-axis. Since the slope of $OC = 3$, then $\tan \theta = 3$, since the tangent of an angle is equal to the slope of a line that makes this angle with the horizontal (the positive $x$-axis in this case).

The angle that $OB$ makes with the positive $x$-axis is $\theta - 45^\circ$. The slope of $OB$ will equal $\tan(\theta - 45^\circ)$.

Using the fact that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$,

$$
\tan(\theta - 45^\circ) = \frac{\tan \theta - \tan 45^\circ}{1 + \tan \theta \tan 45^\circ} = \frac{3 - 1}{1 + 3(1)} = \frac{2}{4} = \frac{1}{2}
$$

Therefore, the slope of $OB = \tan(\theta - 45^\circ) = \frac{1}{2}$. 

Problem of the Week
Problem E
What’s Left?

$BCDE$ is a square with sides of length 20 cm. $BE$ is extended to $A$ such that the area of $\triangle ABC$ is twice the area of the square. The figure $ABCD$ is enclosed in a circle with diameter $AC$ and point $B$ on the circumference of the circle. (The diagram represents the information from the problem but is not necessarily drawn to scale.)

Determine the area inside the circle but outside figure $ABCD$.

NOTE: We know that this circle can be drawn because of a property of circles: The angle inscribed in a semi-circle is $90^\circ$. In this case, $\angle ABC = 90^\circ$ and $AC$ is the diameter.
Problem of the Week
Problem E and Solution
What’s Left?

Problem

$BCDE$ is a square with sides of length 20 cm. $BE$ is extended to $A$ such that the area of $\triangle ABC$ is twice the area of the square. The figure $ABCD\!E$ is enclosed in a circle with diameter $AC$ and point $B$ on the circumference of the circle. See the diagram to the right. Determine the area inside the circle but outside figure $ABCDF$.

Solution

To find the area of the unshaded region, we need to find the area of the circle and subtract the area of the shaded figure $ABCDF$. To find the area of the circle we need the radius, which is half the length of diameter $AC$. To find the area of the shaded figure, we need to find the areas of square $BCDE$ and $\triangle AEF$. We will need to find the length of $EF$.

Area of Square $BCDE = 20 \times 20 = 400 \text{ cm}^2$

Area of $\triangle ABC = 2 \times \text{Area of Square } BCDE = 800 \text{ cm}^2$

But Area of $\triangle ABC = (BC)(AB) \div 2$

$\therefore 800 = (20)(AB) \div 2$

$AB = 80 \text{ cm}$

Then $AE = AB - BE = 80 - 20 = 60 \text{ cm}$.

Since $\angle AEF = \angle ABC = 90^\circ$ and $\angle FAE = \angle CAB$, then $\triangle AEF \sim \triangle ABC$.

$\therefore \frac{AE}{AB} = \frac{EF}{BC}$

$\quad \frac{60}{80} = \frac{EF}{20}$

$EF = 15 \text{ cm}$

Since $\triangle ABC$ is right angled, $AC^2 = BC^2 + AB^2$

$\quad = 20^2 + 80^2$

$\quad = 6800$

$AC = 20\sqrt{17} \text{ cm}$

But $AC$ is the diameter of the circle so the radius is $10\sqrt{17}$.

Unshaded Area $= \text{Area of Circle} - \text{Area of } ABCDF$

$= \text{Area of Circle} - (\text{Area of Square } BCDE + \text{Area of } \triangle AEF)$

$= \pi(10\sqrt{17})^2 - [(20 \times 20) + (15 \times 60 \div 2)]$

$= 1700\pi - 400 - 450$

$= (1700\pi - 850) \text{ cm}^2$

The area inside the circle but outside the shaded figure is $(1700\pi - 850) \text{ cm}^2$ or approximately $4491 \text{ cm}^2$. 
Problem of the Week

Problem E

Paint Free

A cube has edges of length $n$, where $n$ is a positive integer. Three faces, meeting at a corner, are painted red. The cube is then cut into $n^3$ smaller cubes of unit length. That is, the side lengths of the new cubes are 1 unit.

If exactly 125 of these cubes have no faces painted red, determine the value of $n$. 
Problem of the Week

Problem E and Solution

Paint Free

Problem

A cube has edges of length \( n \), where \( n \) is a positive integer. Three faces, meeting at a corner, are painted red. The cube is then cut into \( n^3 \) smaller cubes of unit length. That is, the side lengths of the new cubes are 1 unit. If exactly 125 of these cubes have no faces painted red, determine the value of \( n \).

Solution

Solution 1

This solution requires no content beyond grade ten. The second solution will use the factor theorem which is generally taught in grade twelve.

In the diagram above, the sides painted red are labelled \( A \), \( B \), and \( C \). We know that there are \( n^3 \) unit cubes. To determine the number of unpainted cubes we can subtract the number of cubes with some red from the total number of cubes. Side \( A \) has dimensions \( n \) by \( n \) by 1 and so contains \( n^2 \) unit cubes with some red. Side \( B \) has dimensions \( n \) by \( (n - 1) \) by 1 and so contains \( n \times (n - 1) \) unit cubes with some red. Side \( C \) has dimensions \( (n - 1) \) by \( (n - 1) \) by 1 and so contains \( (n - 1)(n - 1) \) unit cubes with some red. The number of unpainted cubes is \( n^3 - n^2 - n(n - 1) - (n - 1)(n - 1) \). We can simplify this as follows:

\[
\begin{align*}
n^3 - n^2 - n(n - 1) - (n - 1)(n - 1) &= n^2(n - 1) - n(n - 1) - (n - 1)(n - 1) \\
&= (n - 1)(n^2 - n - (n - 1)) \\
&= (n - 1)(n^2 - 2n + 1)
\end{align*}
\]

Each term contains a common factor of \( (n - 1) \) so the expression simplifies to \( (n - 1)(n^2 - 2n + 1) \). This further simplifies to \( (n - 1)^3 \). If the solver pauses here to think about this, if the unit cubes on side \( A \) then side \( B \) and finally side \( C \) are removed we are left with a cube whose side lengths are \( (n - 1) \) and \( (n - 1)^3 \) unit cubes.

But \( (n - 1)^3 = 125 \), the actual number of unpainted cubes. Taking the cube root, \( n - 1 = 5 \) and \( n = 6 \) follows.
Solution 2
The second solution will use the factor theorem which is generally taught in grade twelve.
This solution picks up from the expression giving us the number of unpainted cubes, setting it equal to 125, the number of unpainted cubes.

\[
\begin{align*}
    n^3 - n^2 - n(n-1) - (n-1)(n-1) &= 125 \\
    n^3 - n^2 - n^2 + n - n^2 + 2n - 1 &= 125 \\
    n^3 - 3n^2 + 3n - 126 &= 0
\end{align*}
\]

Let \( f(n) = n^3 - 3n^2 + 3n - 126 \).
When \( n = 6 \), \( f(6) = 6^3 - 3(6^2) + 3(6) - 126 = 216 - 108 + 18 - 126 = 224 - 224 = 0 \). Since \( f(6) = 0 \), \( (n - 6) \) is a factor of \( f(n) \).
After long division (or synthetic division), \( f(n) = (n - 6)(n^2 + 3n + 21) \).
So \( (n - 6)(n^2 + 3n + 21) = 0 \). \( n^2 + 3n + 21 = 0 \) has no real roots so \( n = 6 \) is the only root.
Therefore the original cube has edges of length 6.
Problem of the Week
Problem E
Overlapping, Right?

In the diagram, $AB$ and $BC$ are straight line segments meeting at $B$ so that $\angle ABC = 90^\circ$. $D$ lies on $AB$, $F$ lies on $BC$ and $E$ is the intersection of $AF$ and $DC$. Also, $AD = 1$, $DB = 2$, $AE = 3$, $BF = 4$ and $EF = 2$.

Determine the length of $CF$. 
Problem of the Week
Problem E and Solution
Overlapping, Right?

Problem
In the diagram, \( AB \) and \( BC \) are straight line segments meeting at \( B \) so that \( \angle ABC = 90^\circ \). \( D \) lies on \( AB \), \( F \) lies on \( BC \) and \( E \) is the intersection of \( AF \) and \( DC \). Also, \( AD = 1, \ DB = 2, AE = 3, BF = 4 \) and \( EF = 2 \). Determine the length of \( CF \).

Solution
Draw a perpendicular from \( E \) to \( BF \).
Let \( P \) be the point where the perpendicular intersects \( BF \).
Let \( CF = a, PF = b \) and \( EP = h \).
We will now proceed with three solutions. The first two solutions depend on this setup. The first uses similar triangles to determine \( CF \). The second solution uses trigonometry. The third solution is totally different.

Solution 1
Since \( EP \) is perpendicular to \( BC \), we know \( \angle EPF = 90^\circ \). Also, \( \angle EFP = \angle AFB \) (same angle). Therefore, \( \triangle AFB \sim \triangle EPF \) (by angle-angle triangle similarity).

From the similarity, \( \frac{AF}{BF} = \frac{EP}{PF} \), so \( \frac{5}{4} = \frac{2}{b} \) or \( b = \frac{8}{5} \). Also, \( \frac{AF}{AB} = \frac{EF}{EP} \), so \( \frac{5}{3} = \frac{2}{h} \) or \( h = \frac{6}{5} \).

Now let’s calculate \( PC \). We know \( \angle EPC = \angle DBC = 90^\circ \) and \( \angle ECP = \angle DCB \) (same angle).

Therefore, \( \triangle DBC \sim \triangle EPC \) (by angle-angle triangle similarity). This tells us \( \frac{DB}{BC} = \frac{EP}{PC} \).
Since \( BC = BF + CF = 4 + a \) and \( PC = PF + CF = \frac{8}{5} + a \), we have

\[
\frac{DB}{BC} = \frac{EP}{PC}
\frac{2}{4+a} = \frac{\frac{6}{5}}{\frac{8}{5} + a}
\frac{16}{5} + 2a = \frac{24}{5} + \frac{6}{5}a
2a = \frac{16}{5} - \frac{6}{5}a
\frac{4}{5}a = \frac{8}{5}
a = 2
\]

Therefore, \( CF = 2 \).
Solution 2

In $\triangle EPF$, $\sin \angle EFP = \frac{h}{2}$.

In $\triangle ABF$, $\sin \angle AFB = \frac{3}{5}$.

Since $\angle AFB = \angle EFP$ (same angle),

$$\sin \angle AFB = \sin \angle EFP$$

$$\frac{3}{5} = \frac{h}{2}$$

$$h = \frac{6}{5}$$

Since $\triangle EFP$ is a right-angled triangle,

$$EP^2 + PF^2 = EF^2$$

$$h^2 + b^2 = 2^2$$

$$\left(\frac{6}{5}\right)^2 + b^2 = 4$$

$$b^2 = 4 - \frac{36}{25}$$

$$b^2 = \frac{64}{25}$$

$$b = \frac{8}{5} \text{ since } b > 0$$

In $\triangle ECP$, $\tan \angle ECP = \frac{EP}{PC} = \frac{h}{a + b} = \frac{\frac{6}{5}}{a + \frac{8}{5}}$ and in $\triangle BCD$, $\tan \angle DCB = \frac{DB}{BC} = \frac{2}{4 + a}$.

Since $\angle ECP = \angle DCB$ (same angle),

$$\tan \angle ECP = \tan \angle DCB$$

$$\frac{\frac{6}{5}}{a + \frac{8}{5}} = \frac{2}{4 + a}$$

$$\frac{24}{5} + \frac{6}{5}a = 2a + \frac{16}{5}$$

$$2a - \frac{6}{5}a = \frac{24}{5} - \frac{16}{5}$$

$$\frac{4}{5}a = \frac{8}{5}$$

$$a = 2$$

Therefore, $CF = 2$. 
Solution 3

Using the given information, position $B$ at the origin, $D$ at $(0, 2)$ since $BD = 2$, $A$ at $(0, 3)$ since $AD = 1$, $F$ at $(4, 0)$ since $BF = 4$, and $C$ on the positive $x$-axis at $(c, 0)$ with $c > 4$. Also note that $E$ is in the first quadrant so $x > 0$ and $y > 0$. Construct $FG \perp BF$ with $G$ at $(4, 2)$ so that $FG = FE = BD = 2$. Since $FG = FE = 2$, we can construct a circle with radius 2, centre $F(4, 0)$ and equation $(x - 4)^2 + y^2 = 4$. This circle intersects the line through $AF$ at $E$.

The line passing through $A$ and $F$ has $y$-intercept 3 and slope $-\frac{3}{4}$, leading to equation $y = -\frac{3}{4}x + 3$. To find the intersection $E$, substitute $y = -\frac{3}{4}x + 3$ for $y$ in $(x - 4)^2 + y^2 = 4$.

\[
(x - 4)^2 + \left(-\frac{3}{4}x + 3\right)^2 = 4
\]
\[
x^2 - 8x + 16 + \frac{9}{16}x^2 - \frac{9}{2}x + 9 = 4 \quad \text{(Expand the left side.)}
\]
\[
16x^2 - 128x + 256 + 9x^2 - 72x + 144 = 64 \quad \text{(Multiply by 16.)}
\]
\[
25x^2 - 200x + 336 = 0 \quad \text{(Simplify.)}
\]
\[
(5x - 12)(5x - 28) = 0 \quad \text{(Factor.)}
\]

It follows that $x = \frac{12}{5}$ or $x = \frac{28}{5}$. Substituting $x = \frac{12}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = \frac{6}{5}$.

Substituting $x = \frac{28}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = -\frac{6}{5}$. But $E$ is in the first quadrant so $y > 0$ and this second possibility is inadmissible. It follows that $E$ has coordinates $\left(\frac{12}{5}, \frac{6}{5}\right)$.

We can now find the equation of the line containing $D$, $E$ and $C$. This line has $y$-intercept 2, slope $-\frac{1}{3}$ and equation $y = -\frac{1}{3}x + 2$. The point $C$ is the $x$-intercept of this line and is found by setting $y = 0$. This leads to $x$-intercept 6 and the point $C$ has coordinates $(6, 0)$. Since $F$ is at $(4, 0)$ and $C$ is at $(6, 0)$, $CF = 2$. (It turns out that $C$ is also on the circle through $E$ and $G$.)
Problem of the Week
Problem E
Getting There

In the diagram, $O$ is the origin and $P$ is the point $(6, -4)$. Many paths exist that can get us from point $O$ to point $P$. Two such paths are shown on the grid.

We define the path length between two points $A$ and $B$ as the minimum length along the grid lines from $A$ to $B$.

The path length from $O$ to $P$ is 10. One such path is shown with solid line segments. A second path, shown with dashed lines, has length 20, which is greater than the minimum length path.

How many points with integer coordinates have a path length of 10 from $O$?
Problem of the Week
Problem E and Solution
Getting There

Problem
We define the path length between two points $A$ and $B$ as the minimum length along the grid lines from $A$ to $B$. In the diagram, $O$ is the origin and $P$ is the point $(6, -4)$. The path length from $O$ to $P$ is 10. How many points with integer coordinates have a path length of 10 from $O$?

Solution
Solution 1
Let $Q(a, b)$ be a point that has path length of 10 from $O$, the origin.

Let’s first assume that $Q$ is on the $x$ or $y$ axis.
The only point along the positive $x$-axis that has path length 10 from the origin is $(10, 0)$.
The only point along the negative $x$-axis that has path length 10 from the origin is $(-10, 0)$.
The only point along the positive $y$-axis that has path length 10 from the origin is $(0, 10)$.
The only point along the negative $y$-axis that has path length 10 from the origin is $(0, -10)$.
Therefore, there are 4 points along the axes that have a path length of 10 from $O$.

Next, let’s assume $a > 0$ and $b > 0$, so $Q$ is in the first quadrant.
Since the path length from $O$ to $Q$ is 10, there must be a path from $O$ to $Q$ that moves a total of $r$ units to the right and $u$ units up (in some order) such that $r + u = 10$. This means that $Q$ is $r$ units to the right of $O$ and $u$ units up from $O$. In other words, $a = r$ and $b = u$, so $a + b = r + u = 10$.

The points $(a, b)$ in the first quadrant that satisfy $a + b = 10$ where $a$ and $b$ are integers are $(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)$. There are 9 such pairs. Therefore, there are 9 points in the first quadrant that have path length of 10 from $O$.

By symmetry, there are 9 points in each of the four quadrants that have path length of 10 from $O$.
In quadrant 2, the points are
\[ (-1, 9), (-2, 8), (-3, 7), (-4, 6), (-5, 5), (-6, 4), (-7, 3), (-8, 2), (-9, 1). \]

In quadrant 3, the points are
\[ (-1, -9), (-2, -8), (-3, -7), (-4, -6), (-5, -5), (-6, -4), (-7, -3), (-8, -2), (-9, -1). \]

In quadrant 4, the points are
\[ (1, -9), (2, -8), (3, -7), (4, -6), (5, -5), (6, -4), (7, -3), (8, -2), (9, -1). \]

Therefore, there are a total of $4 + (4 \times 9) = 40$ points with integer coordinates that have a path length of 10 from $O$. 
Solution 2

We are permitted ten moves to get from the origin to a point by traveling along the grid lines. These moves can be all horizontal (in one direction), all vertical (in one direction), or a combination of horizontal moves (in one direction) with vertical moves (in one direction).

We will examine the cases by picking the number of horizontal moves. Then we will determine the corresponding number of vertical moves and the resulting possible endpoints.

• 0 horizontal moves: Since there are no horizontal moves, there would be 10 vertical moves. There are two possible endpoints, (0, 10), and (0, -10).

• 1 horizontal move: Since there is 1 horizontal move, there would be 9 vertical moves. There are four possible endpoints, (-1, 9), (-1, -9), (1, 9), and (1, -9).

• 2 horizontal moves: Since there are 2 horizontal moves, there would be 8 vertical moves. There are four possible endpoints, (-2, 8), (-2, -8), (2, 8), and (2, -8).

• 3 horizontal moves: Since there are 3 horizontal moves, there would be 7 vertical moves. There are four possible endpoints, (-3, 7), (-3, -7), (3, 7), and (3, -7).

• 4 horizontal moves: Since there are 4 horizontal moves, there would be 6 vertical moves. There are four possible endpoints, (-4, 6), (-4, -6), (4, 6), and (4, -6).

• 5 horizontal moves: Since there are 5 horizontal moves, there would be 5 vertical moves. There are four possible endpoints, (-5, 5), (-5, -5), (5, 5), and (5, -5).

• 6 horizontal moves: Since there are 6 horizontal moves, there would be 4 vertical moves. There are four possible endpoints, (-6, 4), (-6, -4), (6, 4), and (6, -4).

• 7 horizontal moves: Since there are 7 horizontal moves, there would be 3 vertical moves. There are four possible endpoints, (-7, 3), (-7, -3), (7, 3), and (7, -3).

• 8 horizontal moves: Since there are 8 horizontal moves, there would be 2 vertical moves. There are four possible endpoints, (-8, 2), (-8, -2), (8, 2), and (8, -2).

• 9 horizontal moves: Since there are 9 horizontal moves, there would be 1 vertical move. There are four possible endpoints, (-9, 1), (-9, -1), (9, 1), and (9, -1).

• 10 horizontal moves: Since there are 10 horizontal moves, there would be 0 vertical moves. There are two possible endpoints, (-10, 0), and (10, 0).

Therefore, there are a total of $2 + (4 \times 9) + 2 = 40$ points with integer coordinates that have a path length of 10 from $O$. 
Points $A$, $B$ and $D$ lie on the circumference of a circle with centre $C$. 
$\angle CAD = p^\circ$ and $\angle CBD = q^\circ$. Determine the measure of $\angle ACB$ and the measure of $\angle ADB$. What is the relationship between $\angle ACB$ and $\angle ADB$?
Problem of the Week
Problem E and Solution
Can You Relate?

Problem
Points $A$, $B$ and $D$ lie on the circumference of a circle with centre $C$. $\angle CAD = p^\circ$ and $\angle CBD = q^\circ$. Determine the measure of $\angle ACB$ and the measure of $\angle ADB$. What is the relationship between $\angle ACB$ and $\angle ADB$?

Solution
We start by constructing radius $CD$.

$CA$ and $CD$ are both radii of the circle, so $CA = CD$. Then $\triangle CAD$ is isosceles and $\angle CDA = \angle CAD = p^\circ$. Since the angles in a triangle add to $180^\circ$, $\angle ACD = (180 - 2p)^\circ$.

$CB$ and $CD$ are both radii of the circle, so $CB = CD$. Then $\triangle CBD$ is isosceles and $\angle CDB = \angle CBD = q^\circ$. Since the angles in a triangle add to $180^\circ$, $\angle BCD = (180 - 2q)^\circ$.

We will now find the measure of $\angle ADB$ and of $\angle ACB$ in order to determine the relationship.

$$\angle ADB = \angle CDB - \angle CDA$$
$$= (q - p)^\circ$$

$$\angle ACB = \angle ACD - \angle BCD$$
$$= (180 - 2p)^\circ - (180 - 2q)^\circ$$
$$= (2q - 2p)^\circ$$
$$= 2 \times (q - p)^\circ$$
$$= 2 \times \angle ADB$$

$\therefore \angle ACB$ is double the size of $\angle ADB$.

In general, the angle inscribed at the centre of a circle is twice the size of the angle inscribed at the circumference by the same chord. In the following diagram, $\angle ACB$ is inscribed at the centre of the circle by chord $AB$ and $\angle ADB$ is inscribed at the circumference by the same chord. Therefore, $\angle ACB = 2 \angle ADB$. 
Problem of the Week

Problem E

Always True?

In the following diagram, \( PQRS \) is a rectangle with \( PQ = SR \) and \( PS = QR \).

The points \( A, B, C, \) and \( D \) are the midpoints of sides \( PQ, QR, RS, \) and \( SP \), respectively. The point \( E \) is the midpoint of line segment \( AD \).

Show that it is always true that the area of rectangle \( PQRS \) is four times the area of \( \triangle BCE \).
Problem of the Week
Problem E and Solution
Always True?

Problem

$PQRS$ is a rectangle with $PQ = SR$ and $PS = QR$. The points $A$, $B$, $C$, and $D$ are the midpoints of sides $PQ$, $QR$, $RS$, and $SP$, respectively. The point $E$ is the midpoint of line segment $AD$. Show that it is always true that the area of rectangle $PQRS$ is four times the area of $\triangle BCE$.

Solution

Solution 1

Let the length of $PQ$ be $2x$ and the length of $PS$ be $2y$. It follows that

\[ PA = AQ = SC = CR = x \]
and
\[ PD = DS = QB = BR = y \]

The area of rectangle $PQRS = PQ \times PS = (2x)(2y) = 4xy$.

Therefore, we need to show that the area of $\triangle BCE = \frac{1}{4}(4xy) = xy$.

Since $PQRS$ is a rectangle, each of the corner angles is $90^\circ$ and each of the four corner triangles are right triangles. Also, each of the four corner triangles, $\triangle APD$, $\triangle BQA$, $\triangle CRB$, and $\triangle DSC$, has base $x$ and height $y$ so their areas are the same. The total area of these four triangles is $4 \times \left(\frac{xy}{2}\right) = 2xy$.

The length of the hypotenuse of each of the four right triangles is $\sqrt{x^2 + y^2}$, so $AB = BC = CD = DA = \sqrt{x^2 + y^2}$ and $ABCD$ is a rhombus. Then, it follows that $AD \parallel CB$.

The area of rhombus $ABCD$ is equal to the area of $PQRS$ minus the area of the four corner right triangles. So, the area of rhombus $ABCD = 4xy - 2xy = 2xy$.

Let $h$ be the perpendicular distance between $AD$ and $CB$, two opposite parallel sides of the rhombus. The area of rhombus $ABCD = h \times BC$, but the area of rhombus $ABCD = 2xy$, so $h \times BC = 2xy$.

Then the area of $\triangle BCE = \frac{h \times BC}{2} = \frac{2xy}{2} = xy$.

Now, the area of $PQRS = 4xy = 4 \times$ the area of $\triangle BCE$, as required.
Solution 2

In this solution we will use analytic geometry.

Let the length of $PQ$ be $4a$ and the length of $PS$ be $4b$. It follows that $PA = AQ = SC = CR = 2a$ and $PD = DS = QB = BR = 2b$.

Position $S$ at the origin, $R$ along the positive $x$-axis at $(4a, 0)$, and $P$ along the positive $y$-axis at $(0, 4b)$. It then follows that $Q$ is at $(4a, 4b)$. The midpoints are then $A(2a, 4b)$, $B(4a, 2b)$, $C(2a, 0)$, and $D(0, 2b)$. Since $E$ is the midpoint of $AD$ it is located at $(a, 3b)$.

Construct a line segment from $E$ perpendicular to $QR$, intersecting $QR$ at $F(4a, 3b)$. Construct a line segment from $E$ perpendicular to $SR$, intersecting $SR$ at $G(a, 0)$. It is easily shown that $EFRG$ is a rectangle.

\[ \text{Area } \triangle BCE = \text{Area } EFRG - \text{Area } \triangle EFB - \text{Area } \triangle BRC - \text{Area } \triangle EGC \]
\[ = GR \times EG - \frac{EF \times FB}{2} - \frac{BR \times CR}{2} - \frac{EG \times GC}{2} \]
\[ = (3a)(3b) - \frac{(3a)(b)}{2} - \frac{(2b)(2a)}{2} - \frac{(3b)(a)}{2} \]
\[ = \frac{18ab - 3ab - 4ab - 3ab}{2} \]
\[ = \frac{4ab}{2} \]
\[ = 4ab \]

Area $PQRS = PQ \times PS$
\[ = (4a)(4b) \]
\[ = 16ab \]
\[ = 4(4ab) \]
\[ = 4 \times \text{the area of } \triangle BCE, \text{ as required.} \]
Problem of the Week
Problem E
A Square Formation

The eight vertices of a regular octagon are randomly labelled $A, B, C, D, E, F, G,$ and $H$. Each letter is used exactly once. The figures below show three different ways to label the vertices of the octagon.

It can be shown that the shape created by drawing a line between every other vertex in a regular octagon is a square, and that this is the only way to use the vertices of a regular octagon to create a square within the octagon.

In the first labelling example, both $ACEG$ and $BDFH$ are squares, as illustrated below in the two diagrams on the left. These are the only two squares that can be created using that specific labelling of the octagon.

In the second example, both $ACEG$ and $BDFH$ are squares, as illustrated below in the middle two diagrams. These are the only two squares that can be created using that specific labelling of the octagon.

In the third example, both $ABCD$ and $EGHF$ are squares, as illustrated below in the two diagrams on the right. These are the only two squares that can be created using that specific labelling of the octagon.

If the vertices of a regular octagon are randomly labelled $A, B, C, D, E, F, G,$ and $H$ and each letter is used exactly once, what is the probability that $ABCD$ is a square?

That is, what is the probability that the figure formed by connecting the vertex labelled $A$ to the vertex labelled $B$, the vertex labelled $B$ to the vertex labelled $C$, the vertex labelled $C$ to the vertex labelled $D$, and the vertex labelled $D$ to the vertex labelled $A$, is a square?
Problem of the Week
Problem E and Solution
A Square Formation

Problem
The vertices of a regular octagon are randomly labelled $A, B, C, D, E, F, G,$ and $H$ and each letter is used exactly once. What is the probability that $ABCD$ is a square?

Solution
Solution 1
In order to determine the probability, we need to determine the number of ways to label the vertices of the regular octagon so that $ABCD$ forms a square and divide by the total number of ways the regular octagon can be labelled.

First, let’s determine the total number of ways that the vertices of a regular octagon can be labelled $A, B, C, D, E, F, G, H,$ in some order.

Let’s start with the vertex on the top left. There are 8 possible ways to label it (it can be labelled as $A, B, C, D, E, F, G$ or $H$). Moving clockwise, the next vertex can be labelled 7 different ways (it can be assigned any letter other than the letter assigned to the previous vertex). Moving clockwise, the next vertex can be labelled 6 different ways (it can be assigned any letter other than the 2 that have already been used). Moving clockwise, the next vertex can be assigned 5 different letters, and so on. Once we reach the last vertex there will only be 1 letter left, so it can be assigned a letter only 1 way.

Therefore, there are

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40,320$$

different ways to label the regular octagon with the letters $A, B, C, D, E, F, G,$ and $H$ in some order.

Now, let’s determine how many of the 40,320 labellings result in $ABCD$ forming a square.

Let’s suppose vertex $A$ is on the top left corner. Then there are two possible ways to label $B, C$ and $D$ so that $ABCD$ forms a square. They are shown below.
For each of these two cases, how many ways can the remaining 4 vertices be labelled? There are 4 choices for labelling the vertex to the right of \( A \) (it can be assigned \( E, F, G \) or \( H \)). Given the labelling of that vertex, moving clockwise, there are 3 choices for the next vertex, then 2 choices for the next and 1 choice for the last vertex.

Therefore, for each of the cases above, there are \( 4 \times 3 \times 2 \times 1 = 24 \) ways to label the remaining vertices. Therefore, there are \( 24 + 24 = 48 \) ways to label the regular octagon with \( A \) in the top left corner and \( ABCD \) forming a square.

Using a similar argument, we can see that for any vertex that \( A \) can be assigned to, there will be 48 ways to label the regular octagon so that \( ABCD \) forms a square. Since \( A \) can be assigned to 8 different vertices, there are \( 8 \times 48 = 384 \) different ways to label the regular octagon so that \( ABCD \) forms a square.

Therefore, the probability that \( ABCD \) forms a square is \( \frac{384}{40320} = \frac{1}{105} \).

**Solution 2**

The first solution counts the number of ways to create square \( ABCD \) and divides by the total number of possible arrangements. This solution uses a more direct probability argument. Since the square labels \( A, B, C, \) and \( D \) are sequential, the \( A \) can go anywhere.

There is now a \( \frac{2}{7} \) chance that the \( B \) will be placed in a location to create a square (either clockwise or counterclockwise around), given where the \( A \) is.

There is now a \( \frac{1}{6} \) chance that the \( C \) will be placed in the only valid location, across from the \( A \), and given that \( B \) is in an acceptable location.

There is now a \( \frac{1}{5} \) chance that the \( D \) will be placed in the only valid location, given the \( A, B, \) and \( C \) are located appropriately.

The remaining assignments are irrelevant to square \( ABCD \).

By multiplying the probabilities, we obtain the probability that \( ABCD \) forms a square is

\[
\frac{2}{7} \times \frac{1}{6} \times \frac{1}{5} = \frac{2}{210} = \frac{1}{105}.
\]
Problem of the Week
Problem E
Circle, Circle, Circle

$AB$ is a diameter of a circle centred at $O$. A line segment is drawn from a point $C$ on the circumference of the circle to $D$ on $AB$ such that $CD \perp AB$ and $CD = \sqrt{3}$ units. Two circles are drawn on $AB$. One has diameter $AD$ and the other has diameter $DB$.

Determine the area of the shaded region. That is, determine the area outside of the two inner circles but inside the outer circle.

It is known that the angle inscribed in a circle by the diameter is $90^\circ$. In the following diagram, $PQ$ is a diameter and $\angle PRQ$ is inscribed in the circle by diameter $PQ$. Therefore, $\angle PRQ = 90^\circ$. 

Problem of the Week
Problem E and Solution
Circle, Circle, Circle

Problem
$AB$ is a diameter of a circle centred at $O$. A line segment is drawn from a point $C$ on the circumference of the circle to $D$ on $AB$ such that $CD \perp AB$ and $CD = \sqrt{3}$ units. Two circles are drawn on $AB$. One has diameter $AD$ and the other has diameter $DB$. Determine the area of the shaded region. That is, determine the area outside of the two inner circles but inside the outer circle.

Solution
Join $A$ to $C$ and $C$ to $B$. Since $AB$ is a diameter and $\angle ACB$ is inscribed in a circle by that diameter, therefore $\angle ACB = 90^\circ$.

Let the radius of the smaller inside circle be $r$. Then the diameter of the smaller inside circle is $DB = 2r$. Let the radius of the larger inside circle be $R$. Then the diameter of the larger inside circle is $AD = 2R$.

Since $CD \perp AB$, then $\angle ADC = \angle BDC = 90^\circ$. We will use the Pythagorean Theorem in the three triangle $\triangle ADC$, $\triangle BDC$, and $\triangle ACB$, to establish a relationship between $R$ and $r$.

All the information is marked in the following diagram.

In $\triangle ADC$, $AC^2 = AD^2 + CD^2 = (2R)^2 + (\sqrt{3})^2 = 4R^2 + 3$.
In $\triangle BDC$, $CB^2 = BD^2 + CD^2 = (2r)^2 + (\sqrt{3})^2 = 4r^2 + 3$.
In $\triangle ACB$, $AB^2 = BC^2 + AC^2 = (4R^2 + 3) + (4r^2 + 3) = 4R^2 + 4r^2 + 6$.
But $AB^2 = (AD + DB)^2 = (2R + 2r)^2 = (2R + 2r)(2R + 2r) = 4R^2 + 8Rr + 4r^2$.
\[
\therefore 4R^2 + 8Rr + 4r^2 = 4R^2 + 4r^2 + 6 \quad \text{and} \quad 8Rr = 4 \quad \text{or} \quad Rr = \frac{3}{4}
\] follows.
The radius of the smaller inner circle is \( r \), the radius of the larger inner circle is \( R \), and the radius of the outer circle is \( (R + r) \). We can now find the shaded area.

\[
\text{Shaded Area} = \text{Area Outer Circle} - \text{Area Larger Inner Circle} - \text{Area Smaller Inner Circle} \\
= \pi \times (R + r)^2 - \pi \times R^2 - \pi \times r^2 \\
= \pi \times (R^2 + 2Rr + r^2) - \pi R^2 - \pi r^2 \\
= \pi R^2 + 2\pi Rr + \pi r^2 - \pi R^2 - \pi r^2 \\
= 2\pi Rr \\
= 2\pi \times \frac{3}{4}, \text{ since } Rr = \frac{3}{4} \\
= \frac{3\pi}{2}
\]

Therefore, the shaded area is \( \frac{3\pi}{2} \) units\(^2\).

**NOTE:** The relationship \( Rr = \frac{3}{4} \) could also be established using similar triangles as follows:

In \( \triangle ACD \), \( \angle CAD + \angle ACD = 90^\circ \) (1).

Since \( \angle ACB = 90^\circ \), \( \angle ACD + \angle DCB = 90^\circ \) (2).

Subtracting (2) from (1) we get \( \angle CAD - \angle DCB = 0 \). The equation simplifies to \( \angle CAD = \angle DCB \).

Now \( \angle CAD = \angle DCB \) and \( \angle CDA = \angle CDB = 90^\circ \). Therefore, \( \triangle ADC \sim \triangle CDB \). From triangle similarity,

\[
\begin{align*}
\frac{AD}{CD} &= \frac{CD}{DB} \\
2R &= \frac{\sqrt{3}}{2r} \\
\sqrt{3} &= \frac{3}{2r} \\
4Rr &= 3 \\
Rr &= \frac{3}{4}
\end{align*}
\]
The parabola \( y = -x^2 + 4 \) has vertex labelled \( P \) and intersects the \( x \)-axis at the points labelled \( A \) and \( B \). The parabola is translated from its original position so that its vertex moves along the line \( y = x + 4 \) to the point \( Q \). In this new position, the parabola intersects the \( x \)-axis at the points labelled \( B \) and \( C \). Determine the coordinates of \( C \).
Problem of the Week
Problem E and Solution
Parabolic Move

Problem
The parabola $y = -x^2 + 4$ has vertex labelled $P$ and intersects the $x$-axis at the points labelled $A$ and $B$. The parabola is translated from its original position so that its vertex moves along the line $y = x + 4$ to the point $Q$. In this new position, the parabola intersects the $x$-axis at the points labelled $B$ and $C$. Determine the coordinates of $C$.

Solution
For the original parabola $y = -x^2 + 4$, the vertex is $P(0, 4)$ and the $x$-intercepts are $A(-2, 0)$ and $B(2, 0)$.

Let the vertex of the translated parabola be $Q(q, p)$. Since the new parabola is a translation of the original, the equation of this new parabola is $y = -(x - q)^2 + p$.

Since $Q$ lies on the line $y = x + 4$, we have $p = q + 4$ and the equation of the new parabola is $y = -(x - q)^2 + q + 4$.

Since $B(2, 0)$ lies on the new parabola, we can substitute $(2, 0)$ into this equation:

\[
\begin{align*}
0 &= -(2 - q)^2 + q + 4 \\
0 &= -(q^2 - 4q + 4) + q + 4 \\
0 &= -q^2 + 4q - 4 + q + 4 \\
0 &= -q^2 + 5q \\
0 &= -q(q - 5)
\end{align*}
\]

Therefore, $q = 0$ or $q = 5$. The value $q = 0$ corresponds to point $P(0, 4)$ in the original parabola. Therefore, $q = 5$ and the vertex of the new parabola is $(5, 9)$ and the equation of this parabola is $y = -(x - 5)^2 + 9$.

Since $C$ is an $x$-intercept of this parabola, to determine $C$ set $y = 0$ in the equation for the parabola and solve for $x$:

\[
\begin{align*}
0 &= -(x - 5)^2 + 9 \\
(x - 5)^2 &= 9 \\
x - 5 &= \pm 3 \\
x &= 8, 2
\end{align*}
\]

The value $x = 2$ corresponds to point $B$, and the value $x = 8$ corresponds to point $C$. Therefore, the coordinates of $C$ are $(8, 0)$. 
Problem of the Week

Problem E

Cornered

A cube of side length 8 cm is shown with a corner removed by making a cut through the midpoints of three adjacent sides.

Similar cuts are made at each of the remaining corners of the cube.

Determine the increase or decrease in total surface area as a result of slicing the eight corners off the original cube.
Problem
A cube of side length 8 cm is shown above with a corner removed by making a cut through the midpoints of three adjacent sides. Similar cuts are made at each of the remaining corners of the cube.

Determine the increase or decrease in total surface area as a result of slicing the eight corners off the original cube.

Solution
To determine the increase or decrease in surface area, we need to only look at one corner, find the surface area increase or decrease there and then multiply the result by eight to account for the eight corners. At each corner, the surface area of three right-angled triangles (top image to the right) is removed and replaced by the surface area of a single triangle (middle image to the right).

Since the cut is made from the midpoints of three adjacent sides, each right triangle has base of length 4 cm and a height of length 4 cm. The resulting area of one of the right triangles is \( \frac{1}{2}(4)(4) = 8 \text{ cm}^2 \).

The hypotenuse of the right triangle is found using Pythagoras’ Theorem, \( \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2} \text{ cm} \). Since the length of the hypotenuse is the same in each of the three right triangles, the remaining triangle in the corner is equilateral with sides of length \( 4\sqrt{2} \text{ cm} \). The triangle looks like the bottom triangle shown to the right.

The altitude of an equilateral triangle right bisects the base. Let \( h \) be the height of the equilateral triangle. Using Pythagoras’ Theorem, \( h^2 = (4\sqrt{2})^2 - (2\sqrt{2})^2 = 32 - 8 = 24 \). Therefore \( h = \sqrt{24} = 2\sqrt{6} \text{ cm} \) (since \( h > 0 \)).

The area of the remaining triangle is \( \frac{1}{2}(4\sqrt{2})(2\sqrt{6}) = 4\sqrt{12} = 8\sqrt{3} \text{ cm}^2 \).

For each corner, the new surface area is increased by the area of the equilateral triangle and decreased by the areas of the three right triangles. Therefore, slicing off a corner changes the surface area by \( 8\sqrt{3} - 3 \times 8 = (8\sqrt{3} - 24) \text{ cm}^2 \). Since \( 8\sqrt{3} < 24 \), this result is negative and the surface area is decreased in each corner. Therefore, slicing off a corner decreases the surface area by \( (24 - 8\sqrt{3}) \text{ cm}^2 \).

Since there are eight corners, the total decrease in surface area is

\[ 8 \times (24 - 8\sqrt{3}) = 192 - 64\sqrt{3} = 64(3 - \sqrt{3}) \approx 81.1 \text{ cm}^2. \]

Therefore, as a result of cutting off each corner, the total surface area decreases by \( 64(3 - \sqrt{3}) \text{ cm}^2 \).
Number Theory
&
Proportions
Problem of the Week
Problem E
Many Possibilities?

Two distinct positive integers are multiplied together. This product is then added to the sum of the original two integers resulting in a sum of 195.

Determine all possible pairs of integers which satisfy the above conditions.
Problem of the Week
Problem E and Solution
Many Possibilities?

Problem

Two distinct positive integers are multiplied together. This product is then added to the sum of the original two integers resulting in a sum of 195.

Determine all possible pairs of integers which satisfy all of the conditions.

Solution

Let \(x\) and \(y\) represent the two positive integers. Since the integers are distinct, \(x \neq y\).

The product of the two integers is \(xy\) and the sum is \((x + y)\).

We want to find all pairs of integers, \(x, y\), such that \(xy + x + y = 195\).

If we attempt to factor the left side of the equation, we discover that there is no common factor between all three terms. However, if we factor a common factor out of the first two terms on the left side of the equation, we obtain \(x(y + 1) + y = 195\).

If we add 1 to both sides, we obtain:

\[
x(y + 1) + y + 1 = 195 + 1
\]

or \(x(y + 1) + 1(y + 1) = 196\).

Now, the left side has a common factor of \((y + 1)\). After factoring, we obtain:

\[(x + 1)(y + 1) = 196.\]

Looking strictly at the equation \((x + 1)(y + 1) = 196\), we see that we are looking for a pair of integers whose product is 196. Using the factors of 196, we obtain

\[
196 = 1 \times 196 = 2 \times 98 = 4 \times 49 = 7 \times 28 = 14 \times 14.
\]

We could also list the first four products in reverse order but the pairs of numbers producing these products will be the same as the pairs producing the first four products already listed.

The numbers in the products are each one more than the numbers we are looking for.

For the product \(196 = 1 \times 196\), \(x = 0\) and \(y = 195\). Since the required numbers are positive integers, this solution is inadmissible.

For the product \(196 = 2 \times 98\), \(x = 1\) and \(y = 97\). This is a valid solution.

For the product \(196 = 4 \times 49\), \(x = 3\) and \(y = 48\). This is a valid solution.

For the product \(196 = 7 \times 28\), \(x = 6\) and \(y = 27\). This is a valid solution.

For the product \(196 = 14 \times 14\), \(x = 13\) and \(y = 13\). Since the required numbers are distinct positive integers, this solution is inadmissible.

There are three pairs of distinct positive integers for which their product and their sum add to 195. The pairs are 1 and 97, 3 and 48, and 6 and 27. A verification is provided on the next page.
<table>
<thead>
<tr>
<th>First Positive Integer</th>
<th>Second Positive Integer</th>
<th>Product</th>
<th>Sum</th>
<th>Product + Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>97</td>
<td>97</td>
<td>98</td>
<td>97 + 98 = 195</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>144</td>
<td>51</td>
<td>144 + 51 = 195</td>
</tr>
<tr>
<td>6</td>
<td>27</td>
<td>162</td>
<td>33</td>
<td>162 + 33 = 195</td>
</tr>
</tbody>
</table>

Note: if the two positive integers did not need to be distinct, then $x = 13$ and $y = 13$ would also be a valid solution since $13 \times 13 = 169$, $13 + 13 = 26$ and $169 + 26 = 195$.

If the problem asked for integer solutions, the number of solutions would increase still further. It is important to pay attention to any and all restrictions stated or implied in a problem.
Problem of the Week
Problem E
Paint Free

A cube has edges of length \( n \), where \( n \) is a positive integer. Three faces, meeting at a corner, are painted red. The cube is then cut into \( n^3 \) smaller cubes of unit length. That is, the side lengths of the new cubes are 1 unit.

If exactly 125 of these cubes have no faces painted red, determine the value of \( n \).
Problem of the Week
Problem E and Solution
Paint Free

Problem
A cube has edges of length \( n \), where \( n \) is a positive integer. Three faces, meeting at a corner, are painted red. The cube is then cut into \( n^3 \) smaller cubes of unit length. That is, the side lengths of the new cubes are 1 unit. If exactly 125 of these cubes have no faces painted red, determine the value of \( n \).

Solution
Solution 1
This solution requires no content beyond grade ten. The second solution will use the factor theorem which is generally taught in grade twelve.

In the diagram above, the sides painted red are labelled \( A \), \( B \), and \( C \). We know that there are \( n^3 \) unit cubes. To determine the number of unpainted cubes we can subtract the number of cubes with some red from the total number of cubes. Side \( A \) has dimensions \( n \) by \( n \) by 1 and so contains \( n^2 \) unit cubes with some red. Side \( B \) has dimensions \( n \) by \( (n - 1) \) by 1 and so contains \( n \times (n - 1) \) unit cubes with some red. Side \( C \) has dimensions \( (n - 1) \) by \( (n - 1) \) by 1 and so contains \( (n - 1)(n - 1) \) unit cubes with some red. The number of unpainted cubes is \( n^3 - n^2 - n(n - 1) - (n - 1)(n - 1) \). We can simplify this as follows:

\[
 n^3 - n^2 - n(n - 1) - (n - 1)(n - 1) = n^2(n - 1) - n(n - 1) - (n - 1)(n - 1) 
\]

Each term contains a common factor of \((n - 1)\) so the expression simplifies to 
\((n - 1)(n^2 - n - (n - 1)) = (n - 1)(n^2 - 2n + 1)\). This further simplifies to \((n - 1)^3\). If the solver pauses here to think about this, if the unit cubes on side \( A \) then side \( B \) and finally side \( C \) are removed we are left with a cube whose side lengths are \((n - 1)\) and \((n - 1)^3\) unit cubes.

But \((n - 1)^3 = 125\), the actual number of unpainted cubes. Taking the cube root, \( n - 1 = 5 \) and \( n = 6 \) follows.
Solution 2

The second solution will use the factor theorem which is generally taught in grade twelve.

This solution picks up from the expression giving us the number of unpainted cubes, setting it equal to 125, the number of unpainted cubes.

\[
\begin{align*}
n^3 - n^2 - n(n - 1) - (n - 1)(n - 1) &= 125 \\
n^3 - n^2 - n^2 + n - n^2 + 2n - 1 &= 125 \\
n^3 - 3n^2 + 3n - 126 &= 0
\end{align*}
\]

Let \( f(n) = n^3 - 3n^2 + 3n - 126 \).

When \( n = 6 \), \( f(6) = 6^3 - 3(6^2) + 3(6) - 126 = 216 - 108 + 18 - 126 = 224 - 224 = 0 \). Since \( f(6) = 0 \), \((n - 6)\) is a factor of \( f(n) \).

After long division (or synthetic division), \( f(n) = (n - 6)(n^2 + 3n + 21) \).

So \((n - 6)(n^2 + 3n + 21) = 0\). \( n^2 + 3n + 21 = 0 \) has no real roots so \( n = 6 \) is the only root.

Therefore the original cube has edges of length 6.
Problem of the Week
Problem E
Overlapping, Right?

In the diagram, $AB$ and $BC$ are straight line segments meeting at $B$ so that $\angle ABC = 90^\circ$. $D$ lies on $AB$, $F$ lies on $BC$ and $E$ is the intersection of $AF$ and $DC$. Also, $AD = 1$, $DB = 2$, $AE = 3$, $BF = 4$ and $EF = 2$.

Determine the length of $CF$.
Problem of the Week
Problem E and Solution
Overlapping, Right?

Problem
In the diagram, \(AB\) and \(BC\) are straight line segments meeting at \(B\) so that \(\angle ABC = 90^\circ\). \(D\) lies on \(AB\), \(F\) lies on \(BC\) and \(E\) is the intersection of \(AF\) and \(DC\). Also, \(AD = 1\), \(DB = 2\), \(AE = 3\), \(BF = 4\) and \(EF = 2\). Determine the length of \(CF\).

Solution
Draw a perpendicular from \(E\) to \(BF\).
Let \(P\) be the point where the perpendicular intersects \(BF\).
Let \(CF = a\), \(PF = b\) and \(EP = h\).
We will now proceed with three solutions. The first two solutions depend on this setup. The first uses similar triangles to determine \(CF\). The second solution uses trigonometry. The third solution is totally different.

Solution 1
Since \(EP\) is perpendicular to \(BC\), we know \(\angle EPF = 90^\circ\). Also, \(\angle EFP = \angle AFB\) (same angle). Therefore, \(\triangle ABF \sim \triangle EPF\) (by angle-angle triangle similarity).

From the similarity, \(\frac{AF}{BF} = \frac{EF}{PF}\), so \(\frac{5}{4} = \frac{2}{b}\) or \(b = \frac{8}{5}\). Also, \(\frac{AF}{AB} = \frac{EF}{EP}\), so \(\frac{5}{3} = \frac{2}{h}\) or \(h = \frac{6}{5}\).

Now let’s calculate \(PC\). We know \(\angle EPC = \angle DBC = 90^\circ\) and \(\angle ECP = \angle DCB\) (same angle).

Therefore, \(\triangle DBC \sim \triangle EPC\) (by angle-angle triangle similarity). This tells us \(\frac{DB}{BC} = \frac{EP}{PC}\).
Since \(BC = BF + CF = 4 + a\) and \(PC = PF + CF = \frac{8}{5} + a\), we have

\[
\frac{DB}{BC} = \frac{EP}{PC}
\]

\[
\frac{2}{4+a} = \frac{\frac{6}{5}}{\frac{8}{5}+a}
\]

\[
\frac{16}{5} + 2a = \frac{24}{5} + \frac{6}{5}a
\]

\[
2a - \frac{6}{5}a = \frac{24}{5} - \frac{16}{5}
\]

\[
\frac{4}{5}a = \frac{8}{5}
\]

\[
a = 2
\]

Therefore, \(CF = 2\).
Solution 2

In $\triangle EPF$, $\sin \angle EFP = \frac{h}{2}$.

In $\triangle ABF$, $\sin \angle AFB = \frac{3}{5}$.

Since $\angle AFB = \angle EFP$ (same angle),

$$\sin \angle AFB = \sin \angle EFP$$

$$\frac{3}{5} = \frac{h}{2}$$

$$h = \frac{6}{5}$$

Since $\triangle EFP$ is a right-angled triangle,

$$EP^2 + PF^2 = EF^2$$

$$h^2 + b^2 = 2^2$$

$$\left(\frac{6}{5}\right)^2 + b^2 = 4$$

$$b^2 = 4 - \frac{36}{25}$$

$$b^2 = \frac{64}{25}$$

$$b = \frac{8}{5}$$ since $b > 0$

In $\triangle ECP$, $\tan \angle ECP = \frac{EP}{PC} = \frac{h}{a+b} = \frac{6}{5} \frac{8}{5}$ and in $\triangle BCD$, $\tan \angle DCB = \frac{DB}{BC} = \frac{2}{4+a}$.

Since $\angle ECP = \angle DCB$ (same angle),

$$\tan \angle ECP = \tan \angle DCB$$

$$\frac{\frac{6}{5}}{a + \frac{8}{5}} = \frac{2}{4 + a}$$

$$\frac{24}{5} + \frac{6}{5}a = 2a + \frac{16}{5}$$

$$2a - \frac{6}{5}a = \frac{24}{5} - \frac{16}{5}$$

$$\frac{4}{5}a = \frac{8}{5}$$

$$a = 2$$

Therefore, $CF = 2$. 
Solution 3

Using the given information, position $B$ at the origin, $D$ at $(0, 2)$ since $BD = 2$, $A$ at $(0, 3)$ since $AD = 1$, $F$ at $(4, 0)$ since $BF = 4$, and $C$ on the positive $x$-axis at $(c, 0)$ with $c > 4$. Also note that $E$ is in the first quadrant so $x > 0$ and $y > 0$. Construct $FG \perp BF$ with $G$ at $(4, 2)$ so that $FG = FE = BD = 2$. Since $FG = FE = 2$, we can construct a circle with radius 2, centre $F(4, 0)$ and equation $(x - 4)^2 + y^2 = 4$. This circle intersects the line through $AF$ at $E$.

The line passing through $A$ and $F$ has $y$-intercept 3 and slope $-\frac{3}{4}$, leading to equation $y = -\frac{3}{4}x + 3$. To find the intersection $E$, substitute $y = -\frac{3}{4}x + 3$ for $y$ in $(x - 4)^2 + y^2 = 4$.

\[
(x - 4)^2 + \left(-\frac{3}{4}x + 3\right)^2 = 4
\]

\[
x^2 - 8x + 16 + \frac{9}{16}x^2 - \frac{9}{2}x + 9 = 4 \quad \text{(Expand the left side.)}
\]

\[
16x^2 - 128x + 256 + 9x^2 - 72x + 144 = 64 \quad \text{(Multiply by 16.)}
\]

\[
25x^2 - 200x + 336 = 0 \quad \text{(Simplify.)}
\]

\[
(5x - 12)(5x - 28) = 0 \quad \text{(Factor.)}
\]

It follows that $x = \frac{12}{5}$ or $x = \frac{28}{5}$. Substituting $x = \frac{12}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = \frac{6}{5}$.

Substituting $x = \frac{28}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = -\frac{6}{5}$. But $E$ is in the first quadrant so $y > 0$ and this second possibility is inadmissible. It follows that $E$ has coordinates $\left(\frac{12}{5}, \frac{6}{5}\right)$.

We can now find the equation of the line containing $D$, $E$ and $C$. This line has $y$-intercept 2, slope $-\frac{1}{3}$ and equation $y = -\frac{1}{3}x + 2$. The point $C$ is the $x$-intercept of this line and is found by setting $y = 0$. This leads to $x$-intercept 6 and the point $C$ has coordinates $(6, 0)$. Since $F$ is at $(4, 0)$ and $C$ is at $(6, 0)$, $CF = 2$. (It turns out that $C$ is also on the circle through $E$ and $G$.)
Problem of the Week
Problem E
Triple Number Sums

The set \{3, 6, 9, 12, 15, \cdots, 2016, 2019\} contains all of the multiples of three from 3 to 2019.

Three distinct numbers are chosen from the set to form a sum. How many different sums can be formed?
Problem of the Week
Problem E and Solution
Triple Number Sums

Problem
The set \{3, 6, 9, 12, 15, \cdots, 2016, 2019\} contains all of the multiples of three from 3 to 2019. Three distinct numbers are chosen from the set to form a sum. How many different sums can be formed?

Solution
Since the set includes every positive multiple of three from 3 to 2019 and 2019 is the largest number, then there are \(2019 \div 3 = 673\) numbers in the set. Each number is of the form \(3n\), for \(n = 1, 2, 3, \cdots, 673\). The required sum is \(3a + 3b + 3c\) where \(a, b,\) and \(c\) are three distinct numbers chosen from \(\{1, 2, 3, \cdots, 673\}\). But \(3a + 3b + 3c = 3(a + b + c)\). We can reduce the problem to the much easier question of, “How many distinct integers can be formed by adding three numbers from \(\{1, 2, 3, \cdots, 673\}\)?”

Clearly, the smallest number is \(1 + 2 + 3 = 6\) and the largest number is \(671 + 672 + 673 = 2016\). It is reasonably easy to see that it is possible to get every number in between 6 and 2016 by:

a) increasing the sum by replacing a number with one that is 1 larger or,
b) decreasing the sum by replacing a number with one that is 1 smaller.

Therefore, all of the numbers from 6 to 2016 inclusive can be formed. The number of numbers that can be formed is 2011. (Some solvers may think that there are 2010 numbers. There are 2016 integers from 1 to 2016, inclusive. But this includes the five numbers 1 to 5. So there are \(2016 - 5 = 2011\) numbers from 6 to 2016.)

This answer, 2011, is the answer to the original problem as well. If \(a + b + c = 6\) then \(3(a + b + c) = 18\). This is the smallest number that is the sum of the three smallest numbers, 3, 6 and 9, from the original set. If \(a + b + c = 2016\) then \(3(a + b + c) = 6048\). This is the largest number that is the sum of the three largest numbers, 2013, 2016, and 2019, from the original set. Then every multiple of three from 18 to 6048 can be generated by adding three different numbers from the original set. (There are 2011 multiples of three from 18 to 6048, inclusive. And each of these can be obtained by adding three distinct numbers from the original set.)
Angela and Barry share a piece of land. The ratio of the area of Angela’s portion of land to the area of Barry’s portion of land is 3 : 2. They each grow corn and peas on their piece of land. The entire piece of land is covered by corn and peas in the ratio 7 : 3. On Angela’s portion of the land, the ratio of corn to peas is 4 : 1. What is the ratio of corn to peas for Barry’s portion of land?
Problem

Angela and Barry share a piece of land. The ratio of the area of Angela’s portion of land to the area of Barry’s portion of land is $3 : 2$. They each grow corn and peas on their piece of land. The entire piece of land is covered by corn and peas in the ratio $7 : 3$. On Angela’s portion of the land, the ratio of corn to peas is $4 : 1$. What is the ratio of corn to peas for Barry’s portion of land?

Solution

Solution 1

Suppose that Angela and Barry share 100 hectares of land. (We may assume any convenient total area.)

Since the ratio of the area of Angela’s land to the area of Barry’s land is $3 : 2$, then Angela has $\frac{3}{5}$ of the 100 hectares, or 60 hectares. Barry has the remaining 40 hectares.

Since the entire piece of land is covered by corn and peas in the ratio $7 : 3$, then $\frac{7}{10}$ of the 100 hectares (that is, 70 hectares) is covered with corn and the remaining 30 hectares with peas.

On Angela’s land, the ratio of corn to peas is $4 : 1$, so $\frac{4}{5}$ of her 60 hectares, or 48 hectares, is covered with corn and the remaining 12 hectares with peas.

Since there are 70 hectares of corn in total, then Barry has $70 - 48 = 22$ hectares of corn.

Since there are 30 hectares of peas in total, then Barry has $30 - 12 = 18$ hectares of peas.

Therefore, the ratio of corn to peas on Barry’s land is $22 : 18 = 11 : 9$.

Solution 2

Suppose that the total combined area of land is $x$.

Since the ratio of the area of Angela’s land to the area of Barry’s land is $3 : 2$, then Angela has $\frac{3}{5}$ of the land, or $\frac{3}{5}x$, while Barry has the remaining $\frac{2}{5}x$.

Since the entire piece of land is covered by corn and peas in the ratio $7 : 3$, then $\frac{7}{10}x$ is covered with corn and the remaining $\frac{3}{10}x$ with peas.

On Angela’s land, the ratio of corn to peas is $4 : 1$ so $\frac{4}{5}$ of her $\frac{3}{5}x$, or $\frac{4}{5} \left( \frac{3}{5}x \right) = \frac{12}{25}x$, is covered with corn and the remaining $\frac{3}{5}x - \frac{12}{25}x = \frac{3}{25}x$ with peas.

Since the area of corn is $\frac{7}{10}x$ in total, then Barry’s area of corn is $\frac{7}{10}x - \frac{12}{25}x = \frac{11}{50}x$.

Since the area of peas is $\frac{3}{10}x$ in total, then Barry’s area of peas is $\frac{3}{10}x - \frac{3}{25}x = \frac{9}{50}x$.

Therefore, the ratio of corn to peas on Barry’s land is $\frac{11}{50} : \frac{9}{50}x = 11 : 9$. 
Problem of the Week
Problem E
Check Your Units

The sum of the first \( n \) positive integers is \( 1 + 2 + 3 + \ldots + n \).
We define \( a_n \) to be the units digit of the sum of the first \( n \) positive integers.
For example,

\[
\begin{align*}
1 &= 1 \quad \text{and} \quad a_1 = 1, \\
1 + 2 &= 3 \quad \text{and} \quad a_2 = 3, \\
1 + 2 + 3 &= 6 \quad \text{and} \quad a_3 = 6, \\
1 + 2 + 3 + 4 &= 10 \quad \text{and} \quad a_4 = 0, \\
1 + 2 + 3 + 4 + 5 &= 15 \quad \text{and} \quad a_5 = 5.
\end{align*}
\]

Thus, \( a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 3 + 6 + 0 + 5 = 15 \).

Determine the smallest value of \( n \) such that \( a_1 + a_2 + a_3 + \ldots + a_n \geq 2019 \).
Problem of the Week
Problem E and Solution
Check Your Units

Problem
The sum of the first \( n \) positive integers is \( 1 + 2 + 3 + \ldots + n \). We define \( a_n \) to be the units digit of the sum of the first \( n \) positive integers.

Determine the smallest value of \( n \) such that \( a_1 + a_2 + a_3 + \ldots + a_n \geq 2019 \).

Solution
Let’s start by examining the values of \( a_n \) until we start to see a pattern.

\[
\begin{align*}
a_1 &= 1 \\
a_2 &= 3, \text{ since } 1 + 2 = 3 \\
a_3 &= 6, \text{ since } 1 + 2 + 3 = 6 \\
a_4 &= 0, \text{ since } 1 + 2 + 3 + 4 = 10 \\
a_5 &= 5, \text{ since } 10 + 5 = 15 \\
a_6 &= 1, \text{ since } 15 + 6 = 21
\end{align*}
\]

Unfortunately, we do not have a pattern yet. Since \( 21 + 7 = 28 \), \( a_7 = 8 \). We need to keep calculating values of \( a_n \).

Notice that we can determine the units digit of the sum of the first \( n \) integers from the units digit from the sum of the first \( n - 1 \) integers and the units digit of \( n \). For example, to calculate \( a_7 \), we simply need to know that \( a_6 = 1 \) and the sum \( 1 + 7 = 8 \) has units digit 8.

\[
\begin{align*}
a_7 &= 8 \\
a_8 &= 6, \text{ since } a_7 + 8 = 16 \\
a_9 &= 5, \text{ since } a_8 + 9 = 15 \\
a_{10} &= 5, \text{ since } a_9 + 0 = 5 \\
a_{11} &= 6, \text{ since } a_{10} + 1 = 6 \\
a_{12} &= 8, \text{ since } a_{11} + 2 = 8 \\
a_{13} &= 1, \text{ since } a_{12} + 3 = 11 \\
a_{14} &= 5, \text{ since } a_{13} + 4 = 5 \\
a_{15} &= 0, \text{ since } a_{14} + 5 = 10 \\
a_{16} &= 6, \text{ since } a_{15} + 6 = 6 \\
a_{17} &= 3, \text{ since } a_{16} + 7 = 13 \\
a_{18} &= 1, \text{ since } a_{17} + 8 = 11 \\
a_{19} &= 0, \text{ since } a_{18} + 9 = 10 \\
a_{20} &= 0, \text{ since } a_{19} + 0 = 0 \\
a_{21} &= 1, \text{ since } a_{20} + 1 = 1
\end{align*}
\]
The values of $a_n$ should repeat now. Can you see why?

Since $a_{21} = a_1$ and the units digit of 22 equals the units digit of 2, $a_{22} = a_2$. Similarly, since $a_{22} = a_2$ and the units digit of 23 equals the units digit of 3, $a_{23} = a_3$.

We will also have $a_{24} = a_4$, and so on.

Therefore, the values of $a_n$ will repeat every 20 values of $n$.

We can calculate

\[
\begin{align*}
    a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + \\
    a_{16} + a_{17} + a_{18} + a_{19} + a_{20} \\
    = 1 + 3 + 6 + 0 + 5 + 1 + 8 + 6 + 5 + 5 + 6 + 8 + 1 + 5 + 0 + 6 + 3 + 1 + 0 + 0 \\
    = 70
\end{align*}
\]

Since the values of $a_n$ repeat every 20 values of $n$, it is also true that

\[
\begin{align*}
    a_{21} + a_{22} + a_{23} + \ldots + a_{39} + a_{40} &= 70, \\
    a_{41} + a_{42} + a_{43} + \ldots + a_{59} + a_{60} &= 70, \\
    &\text{and so on.}
\end{align*}
\]

Since \( \frac{2019}{70} = 28 \frac{59}{70} \), there are 28 complete cycles of the 20 repeating values of $a_n$.

Therefore, the sum of the first 28 \times 20 = 560 values of $a_n$ sum to 28 \times 70 = 1960.

In other words, $a_1 + a_2 + a_3 + \ldots + a_{559} + a_{560} = 1960$.

Let’s keep adding values of $a_n$ until we reach 2019.

\[
\begin{align*}
    a_{561} + a_{562} + a_{563} + a_{564} + a_{565} + a_{566} + a_{567} + a_{568} + a_{569} + a_{570} \\
    = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} \\
    = 1 + 3 + 6 + 0 + 5 + 1 + 8 + 6 + 5 + 5 \\
    = 40.
\end{align*}
\]

Therefore, $a_1 + a_2 + a_3 \ldots a_{569} + a_{570} = 1960 + 40 = 2000$.

$a_{571} = a_{11} = 6$, $a_{572} = a_{12} = 8$, $a_{573} = a_{13} = 1$ and $a_{574} = a_{14} = 5$.

Now,

\[
\begin{align*}
    a_1 + a_2 + a_3 + \ldots + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} &= 2000 + 6 + 8 + 1 = 2015 \leq 2019.
\end{align*}
\]

and

\[
\begin{align*}
    a_1 + a_2 + a_3 + \ldots + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} + a_{574} &= 2015 + 5 = 2020 \geq 2019.
\end{align*}
\]

Therefore, the smallest value of $n$ such that $a_1 + a_2 + a_3 + \ldots + a_n \geq 2019$ is $n = 574$. 

Problem of the Week  
Problem E  
Only Four?

A three-digit number, \( abc \), is formed so that each of the digits in the number, \( a, b, \) and \( c \), are different. Four of these three-digit numbers have the property that the product of the digits of the original number is the same as the two-digit number formed by the tens digit and units digit of the original number, namely \( bc \).

One of the four numbers \( abc \) is 236, since \( a \times b \times c = 2 \times 3 \times 6 = 36 \), which is \( bc \).

\[
2 \times 3 \times 6 = 36
\]

Find the remaining three numbers.

It may be helpful to recall that any two-digit number of the form \( bc \) can be represented by the sum \( 10b + c \). For example, \( 12 = 10(1) + 2 \).
Problem of the Week
Problem E and Solution
Only Four?

Problem
A three-digit number, \(abc\), is formed so that each of the digits in the number, \(a\), \(b\), and \(c\), are different. Four of these three-digit numbers have the property that the product of the digits of the original number is the same as the two-digit number formed by the tens digit and units digit of the original number, namely \(bc\).

One of the four numbers \(abc\) is 236, since \(a \times b \times c = 2 \times 3 \times 6 = 36\), which is \(bc\).

\[
2 \times 3 \times 6 = \boxed{36}
\]

Find the remaining three numbers.

Solution
We know that there are only 4 answers, so we could attempt a trial and error approach to find the remaining 3 numbers. We will present a more systematic approach.

First, notice that since \(a\), \(b\) and \(c\) are digits and must, therefore, be positive integers between 0 and 9. Also, since the product \(a \times b \times c\) is a two-digit number, none of \(a\), \(b\) or \(c\) can equal zero.

We are asked to find all three-digit numbers \(abc\) such that \(a \times b \times c = 10b + c\).

Since \(b \neq 0\), we can divide by \(b\) and the problem becomes equivalent to finding all integers \(a\), \(b\), \(c\) with \(1 \leq a, b, c \leq 9\) and \(a, b\) and \(c\) distinct such that

\[
a \times c = 10 + \frac{c}{b}
\]

Since \(a\) and \(c\) are integers, then so is \(a \times c\) and we must have that \(\frac{c}{b}\) is an integer as well. Therefore, for each possible value of \(c\), we must have that \(b\) divides exactly into \(c\).

We will break the problem into cases based on the value of \(c\). and then sub-cases based on what the value of \(b\) can be for that particular \(c\).

Case 1: \(c = 1\)
There are no values of \(b\) where \(\frac{c}{b}\) is an integer and \(b \neq c\).

Case 2: \(c = 2\)
For \(\frac{c}{b}\) to be an integer and \(b \neq c\), we must have \(b = 1\). If \(b = 1\) and \(c = 2\),
\(a \times b \times c = 10b + c\) becomes \(2a = 12\) and so \(a = 6\). Therefore, one of the three-digit numbers is 612.
Case 3: $c = 3$

For $\frac{c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$. If $b = 1$ and $c = 3$, $a \times b \times c = 10b + c$ becomes $3a = 13$ and so $a = \frac{13}{3}$. Since $a$ is not an integer, there is no solution in this case.

Case 4: $c = 4$

For $\frac{c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$ or $b = 2$.

Case i: $b = 1$

In this case, $a \times b \times c = 10b + c$ becomes $4a = 14$ and so $a = \frac{7}{2}$. Since $a$ is not an integer, there is no solution in this case.

Case ii: $b = 2$

In this case, $a \times b \times c = 10b + c$ becomes $8a = 24$ and so $a = 3$. Therefore, one of the three-digit numbers is 324.

Case 5: $c = 5$

For $\frac{c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$. If $b = 1$ and $c = 5$, $a \times b \times c = 10b + c$ becomes $5a = 15$ and so $a = 3$. Therefore, one of the three-digit numbers is 315.

Case 6: $c = 6$

For $\frac{c}{b}$ to be an integer and $b \neq c$, we must have $b = 1$, $b = 2$ or $b = 3$.

Case i: $b = 1$

In this case, $a \times b \times c = 10b + c$ becomes $6a = 16$ and so $a = \frac{8}{3}$. Since $a$ is not an integer, there is no solution in this case.

Case ii: $b = 2$

In this case, $a \times b \times c = 10b + c$ becomes $12a = 26$ and so $a = \frac{13}{6}$. Since $a$ is not an integer, there is no solution in this case.

Case iii: $b = 3$

In this case, $a \times b \times c = 10b + c$ becomes $18a = 36$ and so $a = 2$. Therefore, one of the three-digit numbers is 236. This is the number given in the example.

We can actually stop here since we have found 4 different three-digit numbers that satisfy the conditions outlined in the problem. If we had not been given the number of possible solutions, we would need to continue by checking cases when $c = 7$, $c = 8$ and $c = 9$.

Therefore, the 4 three-digit numbers that satisfy the conditions of the problem are 612, 324, 315, and 236.
The sequence \( \{1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots \} \) is called an alternating sequence since the terms alternate positive and negative. In this particular sequence, the value of any term in an odd position in the sequence is the term number and the value of any term in an even position in the sequence is the term number multiplied by \(-1\).

Let \( S(n) \) be the sum of the first \( n \) terms of the sequence. Then it follows that

\[
\begin{align*}
S(1) = 1 & \quad S(2) = 1 - 2 = -1 \\
S(3) = 1 - 2 + 3 = 2 & \quad S(4) = 1 - 2 + 3 - 4 = -2 \\
S(5) = 1 - 2 + 3 - 4 + 5 = 3 & \quad S(6) = 1 - 2 + 3 - 4 + 5 - 6 = -3 \\
S(7) = 1 - 2 + 3 - 4 + 5 - 6 + 7 = 4 & \quad S(8) = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 = -4 \\
S(9) = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 = 5 & \quad S(10) = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 = -5
\end{align*}
\]

The following are two unproven results from the above sums.

If \( n \) is an even positive integer, i.e., \( n = 2k \) where \( k \) is a positive integer, then
\[
S(n) = S(2k) = -k.
\]
For example, \( S(6) = S(2 \times 3) = -3 \).

If \( n \) is an odd positive integer, i.e., \( n = 2m - 1 \) where \( m \) is a positive integer, then
\[
S(n) = S(2m - 1) = m.
\]
For example, \( S(9) = S(2 \times 5 - 1) = 5 \).

Suppose that \( a \) and \( b \) are any positive integers. Using the two results above, show that 
\[
S(a) + S(b) + S(a + b) = 1
\]
when both \( a \) and \( b \) are odd.

As an extension, show that 
\[
S(a) + S(b) + S(a + b) = 1 \text{ only}
\]
when both \( a \) and \( b \) are odd. As a further extension, prove that the two given results are true.
Problem of the Week
Problem E and Solution
Up, Down, Up, Down, ...

Problem

The sequence \{1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots\} is called an alternating sequence since the terms alternate positive and negative. In this particular sequence, the value of any term in an odd position in the sequence is the term number and the value of any term in an even position in the sequence is the term number multiplied by -1.

Let \( S(n) \) be the sum of the first \( n \) terms of the sequence. Then it follows that

\[
\begin{align*}
S(1) &= 1 \\
S(3) &= 1 - 2 + 3 = 2 \\
S(5) &= 1 - 2 + 3 - 4 + 5 = 3 \\
S(7) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 = 4 \\
S(9) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 = 5 \\
S(2) &= 1 - 2 = -1 \\
S(4) &= 1 - 2 + 3 - 4 = -2 \\
S(6) &= 1 - 2 + 3 - 4 + 5 - 6 = -3 \\
S(8) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 = -4 \\
S(10) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 = -5
\end{align*}
\]

Show that \( S(a) + S(b) + S(a + b) = 1 \) when both \( a \) and \( b \) are odd positive integers.

Solution

Suppose \( a \) and \( b \) are both odd positive integers. Then \( a = 2k - 1 \) and \( b = 2m - 1 \), for some positive integers \( k \) and \( m \). If \( k = m \), then \( a = b \), but we want to prove the result true for any odd positive integers \( a \) and \( b \), so it may be the case that \( k \neq m \).

We have \( S(a) = S(2k - 1) = k \) and \( S(b) = S(2m - 1) = m \).

Also, \( a + b = (2k - 1) + (2m - 1) = 2k + 2m - 2 = 2(k + m - 1) \) is even, so

\( S(a + b) = S(2(k + m - 1)) = -(k + m - 1) = -k - m + 1 \).

Then, \( S(a) + S(b) + S(a + b) = k + m + (-k - m + 1) = 1 \).

Therefore, \( S(a) + S(b) + S(a + b) = 1 \) whenever \( a \) and \( b \) are odd positive integers.

Extension

Since we have formulas for \( S(a) \) and \( S(b) \) based on whether \( a \) and \( b \) are even or odd, we will break into cases based on the parity of \( a \) and \( b \) to show that none of the other possible cases make \( S(a) + S(b) + S(a + b) = 1 \) true.

Case 1: \( a \) and \( b \) are both even positive integers

Let \( a = 2k \) and \( b = 2m \), for some positive integers \( k \) and \( m \).

We have \( S(a) = S(2k) = -k \) and \( S(b) = S(2m) = -m \).

Also, \( a + b = 2k + 2m = 2(k + m) \) is even, so \( S(a + b) = S(2(k + m)) = -(k + m) = -k - m \).

Then \( S(a) + S(b) + S(a + b) = -k + m + (-k - m) = -2k - 2m = -2(k + m) \). Since \( k \) and \( m \) are positive integers, \( k + m \) is also a positive integer, so \(-2(k + m)\) is a negative integer and so we cannot have \(-2(k + m) = 1\). Therefore, \( S(a) + S(b) + S(a + b) \neq 1 \).
Case 2: $a$ is even and $b$ is odd

Let $a = 2k$ and $b = 2m - 1$, for some positive integers $k$ and $m$.

We have $S(a) = S(2k) = -k$ and $S(b) = S(2m - 1) = m$.

Also, $a + b = 2k + (2m - 1) = 2(k + m) - 1$ is odd, so $S(a + b) = S(2(k + m) - 1) = k + m$.

Then $S(a) + S(b) + S(a + b) = -k + m + (k + m) = 2m$. Since $m$ is an integer, we cannot have $2m = 1$ (then $m = \frac{1}{2}$). Therefore, $S(a) + S(b) + S(a + b) \neq 1$.

Case 3: $a$ is odd and $b$ is even

Let $a = 2k - 1$ and $b = 2m$, for some positive integers $k$ and $m$.

We have $S(a) = S(2k - 1) = k$ and $S(b) = S(2m) = -m$.

Also, $a + b = (2k - 1) + 2m = 2(k + m) - 1$ is odd, so $S(a + b) = S(2(k + m) - 1) = k + m$.

Then $S(a) + S(b) + S(a + b) = k - m + (k + m) = 2k$. Since $k$ is an integer, we cannot have $2k = 1$ (then $k = \frac{1}{2}$). Therefore, $S(a) + S(b) + S(a + b) \neq 1$.

Therefore, $S(a) + S(b) + S(a + b) = 1$ is only true when both $a$ and $b$ are odd positive integers.

We will now prove the first fact given in the question.

We will show that for numbers of the form $n = 2k$, we have $S(n) = S(2k) = -k$.

Numbers of the form $n = 2k$:

$S(2k) = 1 - 2 + 3 - 4 + 5 + \ldots + (2k - 1) - 2k$

$= (1 + 3 + 5 + \ldots + (2k - 1)) - (2 + 4 + 6 + \ldots + 2k)$

$= (1 + 3 + 5 + \ldots + (2k - 1)) - 2(1 + 2 + 3 + \ldots + k)$

$= (1 + 3 + 5 + \ldots + (2k - 1)) + (2 + 4 + 6 + \ldots + 2k) - (2 + 4 + 6 + \ldots + 2k) - 2(1 + 2 + 3 + \ldots + k)$

$= (1 + 2 + 3 + 4 + 5 + \ldots + (2k - 1) + 2k) - 2(1 + 2 + 3 + \ldots + k) - 2(1 + 2 + 3 + \ldots + k)$

$= (1 + 2 + 3 + 4 + 5 + \ldots + (2k - 1) + 2k) - 4(1 + 2 + 3 + \ldots + k)$

Now, using the formula for the sum of the first $n$ integers, that is

$1 + 2 + 3 + \ldots + (n - 1) + n = \frac{n(n + 1)}{2}$

we have

$S(2k) = (1 + 2 + 3 + 4 + 5 + \ldots + (2k - 1) + 2k) - 4(1 + 2 + 3 + \ldots + k)$

$= \frac{(2k)(2k + 1)}{2} - 4\left(\frac{k(k + 1)}{2}\right)$

$= k(2k + 1) - 2k(k + 1)$

$= 2k^2 + k - 2k^2 - 2k$

$= -k$

The proof that $S(n) = S(2k - 1) = k$ for numbers of the form $n = 2k - 1$ is similar.

We’ll leave that for you to try on your own.
Problem of the Week
Problem E
Boxed In

CEMC Parcel is a company that ships their product in boxes with certain restrictions. The company will only ship their product in boxes where

- the length, width, and height, in cm, are all integers,
- the length, width and height are in the ratio 4 : 3 : 5,
- the sum of the length, width and height is at least 100 cm and at most 1000 cm,
- the volume of the box is less than 2 m³.

Determine the dimensions of the smallest box and of the largest box that satisfy CEMC Parcel’s restrictions.
Problem

CEMC Parcel is a company that ships their product in boxes with certain restrictions. The company will only ship their product in boxes where

- the length, width, and height, in cm, are all integers,
- the length, width, and height are in the ratio 4 : 3 : 5,
- the sum of the length, width, and height is at least 100 cm and at most 1000 cm,
- the volume of the box is less than $2\text{ m}^3$.

Determine the dimensions of the smallest box and of the largest box that satisfy CEMC Parcel’s restrictions.

Solution

Since the boxes used by CEMC Parcel have integer side lengths in the ratio 4 : 3 : 5, let $4n$ represent the length of a box in cm, let $3n$ represent the width of a box in cm, and let $5n$ represent the height of a box in cm, where $n$ is an integer.

Furthermore, the sum of the length, width, and height must be at least 100 cm and at most 1000 cm. So it follows that

\[
4n + 3n + 5n \geq 100 \quad \text{and} \quad 4n + 3n + 5n \leq 1000
\]

\[
12n \geq 100 \quad \text{and} \quad 12n \leq 1000
\]

\[
\frac{12n}{12} \geq \frac{100}{12} \quad \text{and} \quad \frac{12n}{12} \leq \frac{1000}{12}
\]

\[
n \geq \frac{100}{12} \quad \text{and} \quad n \leq \frac{1000}{12}
\]

\[
n \geq \frac{25}{3} \quad \text{and} \quad n \leq \frac{250}{3}
\]

There is one other restriction to consider. The volume of a box used by CEMC Parcel is less than $2\text{ m}^3$. To convert from m$^3$ to cm$^3$, note that

\[
1\text{ m}^3 = 1\text{ m} \times 1\text{ m} \times 1\text{ m} = 100\text{ cm} \times 100\text{ cm} \times 100\text{ cm} = 1\ 000\ 000\ \text{cm}^3, \text{ so } 2\text{ m}^3 = 2\ 000\ 000\ \text{cm}^3.
\]

Therefore,

\[
4n(3n)(5n) < 2\ 000\ 000
\]

\[
60n^3 < 2\ 000\ 000
\]

\[
n^3 < \frac{100\ 000}{3}
\]

\[
n < \sqrt[3]{\frac{100\ 000}{3}} \approx 32.2
\]

We also know that $n$ is an integer. Since $n \geq \frac{25}{3}$, then the smallest possible integer value of $n$ is 9. Since $n \leq \frac{250}{3}$ and $n < 32.2$, then the largest possible value of $n$ is 32.

Using the dimensions $4n$, $3n$ and $5n$ with $n = 9$, the smallest box has dimensions 36 cm by 27 cm by 45 cm. Using the dimensions $4n$, $3n$ and $5n$ with $n = 32$, the largest box has dimensions 128 cm by 96 cm by 160 cm. (The largest box has volume approximately $1.97\text{ m}^3$.)