The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Problem of the Week Problems and Solutions 2019-2020

Problem C (Grade 7/8)

Themes

Number Sense (N)
Geometry (G)
Algebra (A)
Data Management (D)
Computational Thinking (C)

(Click on a theme name above to jump to that section)

*The problems in this booklet are organized into themes. A problem often appears in multiple themes.

Number Sense (N)

TAKE ME TO THE COVER

Problem of the Week

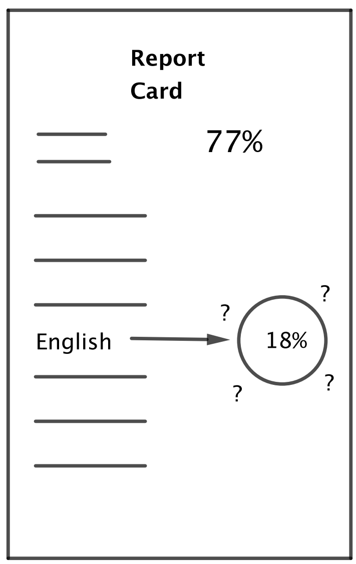
Problem C

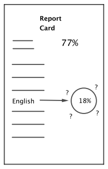
A Reporting Problem

Mildred has seven grades on her report card. The overall average of these seven grades is 77%.

After looking more closely at her report card, Mildred discovered that her English grade was incorrectly recorded as 18% instead of her actual grade of 81%.

Determine Mildred's correct report card average.





Problem of the Week

Problem C and Solution

A Reporting Problem

Problem

Mildred has seven grades on her report card. The overall average of these seven grades is 77%. After looking more closely at her report card, Mildred discovered that her English grade was incorrectly recorded as 18% instead of her actual grade of 81%. Determine Mildred's correct report card average.

Solution

Solution 1

This first solution works primarily with the definition of an average.

To calculate an average we add the seven grades and divide by 7.

$$\frac{\text{sum of seven grades}}{7} = 77$$

To then obtain the sum of the seven grades we would multiply the average by 7.

$$\text{sum of seven grades} = 7 \times 77 = 539$$

But this sum includes the wrong English grade of 18%. So we need to adjust the sum by subtracting the wrong grade and adding the corrected grade.

$$\text{correct sum of seven grades} = 539 - 18 + 81 = 602$$

We can now obtain Mildred's corrected average by dividing the corrected sum by 7.

$$\text{correct average} = \frac{\text{correct sum of seven grades}}{7} = \frac{602}{7} = 86$$

$\therefore$ Mildred's corrected report card average is 86%.

Solution 2

The second solution looks at how an increase in a grade will affect an overall average.
For an average based on seven grades, an increase of 1% for one grade will cause the overall average to increase by $\frac{1}{7}$ of 1%. So, for each increase of 7%, the overall average will increase by 1%. That is, Mildred's mark will increase by 1% for every 7% her English grade increases by.

Mildred's English grade increases by $81\% - 18\% = 63\%$. Since $63 \div 7 = 9$, her average will increase by 9% to 86%.

$\therefore$ Mildred's corrected report card average is 86%.

Problem of the Week

Problem C

It All Adds Up

The *digit sum* of a positive integer is the sum of all of its digits.

For example, the *digit sum* of the integer 1234 is 10, since $1 + 2 + 3 + 4 = 10$.

Find all five-digit positive integers whose *digit sum* is exactly 3.

Theme Number Sense

Problem of the Week
Problem C and Solution
It All Adds Up

Problem

The *digit sum* of a positive integer is the sum of all of its digits. For example, the *digit sum* of the integer 1234 is 10 since $1 + 2 + 3 + 4 = 10$.
Find all five-digit positive integers whose *digit sum* is exactly 3.

Solution

We will find the three groups of five digits that add to 3. We will then rearrange these digits to determine all five-digit positive integers whose digits sum to 3.
Note that the first digit cannot be 0, because then otherwise the integer would not be a five-digit integer. The information is summarized in the following table.

The five digits	The possible five-digit integers	Number of possibilities
$3, 0, 0, 0, 0$	30 000	1
$1, 2, 0, 0, 0$	12 000, 21 000 10 200, 20 100 10 020, 20 010 10 002, 20 001	8
$1, 1, 1, 0, 0$	11 100, 10 110 11 010, 10 101 11 001, 10 011	6

Therefore, the number of five-digit positive integers that have a *digit sum* of 3 is $1 + 8 + 6 = 15$.

Note: It is a known fact that an integer is divisible by 3 exactly when its *digit sum* is divisible by 3. For example, 32 814 has a *digit sum* of $3 + 2 + 8 + 1 + 4 = 18$. Since 18 is divisible by 3, then 32 814 is divisible by 3. On the other hand, 32 810 has a *digit sum* of $3 + 2 + 8 + 1 + 0 = 14$. Since 14 is not divisible by 3, then 32 810 is not divisible by 3.

As a consequence of this fact, each of the 15 five-digit integers we found above must be divisible by 3.

Problem of the Week

Problem C

Vote!

In an election for a grade representative on the school council, there were only two candidates.

Freda First received 60% of the total votes and Saheel Second received all the rest. If Freda won by 28 votes, how many people voted?

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Vote!

Problem

In an election for a grade representative on the school council, there were only two candidates. Freda First received 60% of the total votes and Saheel Second received all the rest. If Freda won by 28 votes, how many people voted?

Solution

If Freda received 60% of the votes, then Saheel received 40% of the total number of votes. The difference between them, 20%, represents 28 votes.

We are interested in determining 100% of the votes, that is, the total number of votes cast. Since we know that 20% of the total votes cast was 28 votes and $5 \times 20 = 100$, then the total number of votes cast was 5×28 or 140 votes.

Solution 2

The second solution uses an algebraic approach.

Let n represent the total number of votes cast.
Since Freda received 60% of the total votes, she received $0.6 \times n$ or $0.6n$ votes.
Since Saheel received all of the remaining votes, he received $0.4 \times n$ or $0.4n$ votes.

We know that the difference between the number of votes received by Freda and the number of votes received by Saheel was 28. So,

$$\begin{aligned} 0.6n - 0.4n &= 28 \\ 0.2n &= 28 \\ \frac{0.2n}{0.2} &= \frac{28}{0.2} \\ n &= 140 \end{aligned}$$

Therefore, there were 140 votes cast.

Problem of the Week

Problem C

Press On

On a game show, contestants are selected to participate in a game in an attempt to win a fabulous prize. One particular game is called "Press On". The game involves a large number pad and the first ten digits of the 12-digit serial number of an 85 inch television. The first ten digits of the serial number are displayed at the top of the number pad. Contestants must guess the last two digits of the serial number.

The contestant is provided with the following information about the serial number:

- No two adjacent digits in the serial number are the same.
- On the keypad, each digit in serial number somehow touches the next digit in the serial number. For example, the digit 1 on the keypad touches digits 2, 4 and 5. The digit 5 on the keypad touches every digit but 0.
- The final three digits in the serial number form a three-digit number that is not divisible by 2, 3 or 5.

When the contestant is ready, they would press their two numbers on the number pad followed by the # key. If the contestant is correct, they would win the television. If they are incorrect, they leave the show with nothing. The contestant has only one chance to select the final two digits of the winning serial number.

If the contestant uses the given information correctly, how many possible 2-digit choices are there for ending the serial number?

159 080 741 4 __ __		
1	2	3
4	5	6
7	8	9
*	0	#

It may be helpful to note that a number is divisible by 3 exactly when the sum of its digits is divisible by 3.

Problem of the Week

Problem C and Solution

Press On

Problem

On a game show, contestants are selected to participate in a game in an attempt to win a fabulous prize. One particular game is called "Press On". The game involves a large number pad and the first ten digits of the 12-digit serial number of an 85 inch television. The first ten digits of the serial number are displayed at the top of the number pad. Contestants must guess the last two digits of the serial number.

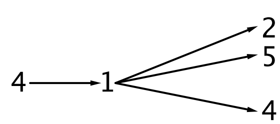
The contestant is provided with the following information about the serial number: no two adjacent digits in the serial number are the same; on the keypad, each digit in serial number somehow touches the next digit in the serial number; and the final three digits in the serial number form a three-digit number that is not divisible by 2, 3 or 5.

When the contestant is ready, they would press their two numbers on the number pad followed by the # key. If the contestant is correct, they would win the television. If they are incorrect, they leave the show with nothing. The contestant has only one chance to select the final two digits of the winning serial number. If the contestant uses the given information correctly, how many possible 2-digit choices are there for ending the serial number?

159 080 741 4 __ __		
1	2	3
4	5	6
7	8	9
*	0	#

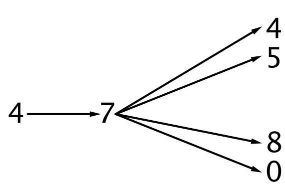
Solution

There are five numbers that touch the 4 on the key pad. Therefore, 1, 2, 5, 7, and 8 are the only possibilities for the first missing digit. We will examine each of the possibilities.



1. First Missing Digit is 1

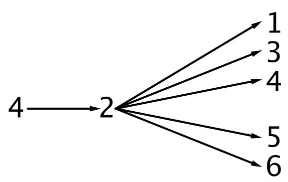
The diagram to the left shows every possibility for the final digit. Since the serial number is not divisible by 2, we can eliminate any of the possibilities with an even final digit. That is, we can eliminate 2 and 4. Since the serial number is not divisible by 5, we can also eliminate 5 as a possible final digit. Since we have eliminated all possible choices for a final digit when the first missing digit is 1, no serial number exists whose first missing digit is 1.



2. First Missing Digit is 7

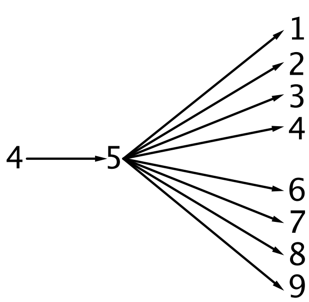
The diagram to the left shows every possibility for the final digit. Since the serial number is not divisible by 2, we can eliminate any of the possibilities with an even final digit. That is, we can eliminate 4, 8 and 0. Since the serial number is not divisible by 5, we can also eliminate 5 as a possible final digit. Since we have eliminated all possible choices for a final digit when the first missing digit is 7, no serial number exists whose first missing digit is 7.

3. First Missing Digit is 2



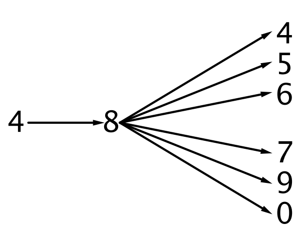
The diagram to the left shows every possibility for the final digit. Since the serial number is not divisible by 2, we can eliminate any of the possibilities with an even final digit. That is, we can eliminate 4 and 6. Since the serial number is not divisible by 5, we can also eliminate 5 as a possible final digit. The remaining possible last three digits are 421 and 423. The sum of the digits of 421 is 7. Since 7 is not divisible by 3, then 421 is not divisible by 3. Therefore, 21 is a valid possibility for the final two digits of the serial number. The sum of the digits of 423 is 9. Since 9 is divisible by 3, then 423 is divisible by 3. Therefore, 23 is not a valid possibility for the final two digits of the serial number.

4. First Missing Digit is 5



The diagram to the left shows every possibility for the final digit. Since the serial number is not divisible by 2, we can eliminate any of the possibilities with an even final digit. That is, we can eliminate 2, 4, 6, and 8. The remaining possible last three digits are 451, 453, 457 and 459. The sum of the digits of 451 and 457 are 10 and 16, respectively. Since neither 10 nor 16 are divisible by 3, then both 451 and 457 are not divisible by 3. Therefore, both 51 and 57 are valid possibilities for the final two digits of the serial number. The sum of the digits of 453 and 459 are 12 and 18, respectively. Since both 12 and 18 are divisible by 3, then both 453 and 459 are divisible by 3. Therefore, 53 or 59 are not valid possibilities for the final two digits of the serial number.

5. First Missing Digit is 8



The diagram to the left shows every possibility for the final digit. Since the serial number is not divisible by 2, we can eliminate any of the possibilities with an even final digit. That is, we can eliminate 4, 6, and 0. Since the serial number is not divisible by 5, we can also eliminate 5 as a possible final digit. The remaining possible last three digits are 487 and 489. The sum of the digits of 487 is 19. Since 19 is not divisible by 3, then 487 is not divisible by 3 and 87 is a valid possibility for the final two digits of the serial number. The sum of the digits of 489 is 21. Since 21 is divisible by 3, then 489 is divisible by 3 and 89 is not a valid possibility for the final two digits of the serial number.

We have examined all of the possibilities for the final two digits of the serial number that satisfy the given information. There are four valid possibilities for the final two digits, namely 21, 51, 57, and 87. Therefore, the contestant has a probability of $\frac{1}{4}$ of selecting the correct final two digits of the serial number. In other words, there is a 25% chance of winning this game.

Problem of the Week
Problem C
Award for Innovation

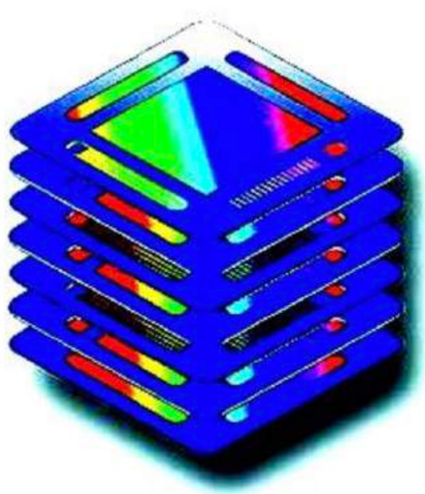
Three co-workers, Arden, Dale and Ryan, received an innovation award for their ground breaking work on battery technology. They have decided to split the award as follows:

(i) Arden receives \$10 000 plus $\frac{1}{5}$ of what then remains;

(ii) Dale then receives \$16 000 plus $\frac{1}{4}$ of what then remains; and

(iii) Ryan then receives the rest, which is \$18 000.

How much is the original monetary award? Which co-worker receives the most money?



Theme Number Sense

Problem of the Week
Problem C and Solution
Award for Innovation

Problem

Three co-workers, Arden, Dale and Ryan, received an innovation award for their ground breaking work on battery technology. They have decided to split the award as follows: Arden receives \$10 000 plus $\frac{1}{5}$ of what then remains; Dale then receives \$16 000 plus $\frac{1}{4}$ of what then remains; and Ryan then receives the rest, which is \$18 000. How much is the original monetary award? Which co-worker receives the most money?

Solution

We will start from Ryan and work towards Arden.

Dale receives $\frac{1}{4}$ of the remainder, so what is left for Ryan is $\frac{3}{4}$ of the remainder.
It follows that \$18 000 is $\frac{3}{4}$ of the remainder.

If $\frac{3}{4}$ of the remainder is \$18 000, then $\frac{1}{4}$ of the remainder is $\$18\,000 \div 3 = \$6000$.

So just after Dale receives \$16 000, there is $\$6000 + \$18\,000$ or \$24 000 left.
Therefore, before Dale gets any money there is $\$24\,000 + \$16\,000$ or \$40 000.
Dale receives $\$16\,000 + \$6000 = \$22\,000$.

Arden receives $\frac{1}{5}$ of the remainder, so what is left for Dale is $\frac{4}{5}$ of the remainder.
It follows that \$40 000 is $\frac{4}{5}$ of the remainder.

If $\frac{4}{5}$ of the remainder is \$40 000, then $\frac{1}{5}$ of the remainder is $\$40\,000 \div 4 = \$10\,000$.

So just after Arden receives \$10 000, there is $\$10\,000 + \$40\,000$ or \$50 000 left.
Therefore, before Arden gets any money there is $\$50\,000 + \$10\,000$ or \$60 000.
Arden receives $\$10\,000 + \$10\,000 = \$20\,000$.

Arden receives \$20 000, Dale receives \$22 000, and Ryan receives \$18 000.

Therefore, the original monetary award was \$60 000 and Dale receives the largest share.

We can check our result by working through the given information one step at a time.

$\rightarrow$ $\$60\,000 - \$10\,000 = \$50\,000$ left after Arden's initial amount.
$\rightarrow$ $\$50\,000 - \frac{1}{5} \times \$50\,000 = \$50\,000 - \$10\,000 = \$40\,000$ remaining after Arden's full share.

$\rightarrow$ $\$40\,000 - \$16\,000 = \$24\,000$ after Dale's initial amount.
$\rightarrow$ $\$24\,000 - \frac{1}{4} \times \$24\,000 = \$24\,000 - \$6000 = \$18\,000$ remaining after Arden and Dale's shares.

$\rightarrow$ \$18 000 is Ryan's share.

Problem of the Week

Problem C

Thinking About Products

Three cards are lined up on a table. Each card has a letter printed on one side and a positive number printed on the other side. One card has an R printed on it, one card has a G printed on it, and one card has a B printed on it. The number side of each card is facedown on the table. The following is known about the three concealed numbers:

(i) the product of the number on the card with an R and the number on the card with a G equals the number on the card with a B;

(ii) the product of the number on the card with a G and the number on the card with a B is 180; and

(iii) five times the number on the card with a B equals the number on the card with a G.

Determine the product of the numbers on the three cards.

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Thinking About Products

Problem

Three cards are lined up on a table. Each card has a letter printed on one side and a positive number printed on the other side. One card has an R printed on it, one card has a G printed on it, and one card has a B printed on it. The number side of each card is facedown on the table. The product of the number on the card with an R and the number on the card with a G equals the number on the card with a B. The product of the number on the card with a G and the number on the card with a B is 180. And, five times the number on the card with a B equals the number on the card with a G. Determine the product of the numbers on the three cards.

Solution

Solution 1

Let the three numbers be represented by r, g, and b.

Since the product of the number on the card with an R and the number on the card with a G equals the number on the card with a B, $r \times g = b$. We are looking for $r \times g \times b = (r \times g) \times b = (b) \times b = b^2$. So when we find b^2 we have found the required product $r \times g \times b$.

We are also given that that $g \times b = 180$ and $g = 5 \times b$, so $g \times b = 180$ becomes $(5 \times b) \times b = 180$ or $5 \times b^2 = 180$. Dividing by 5, we obtain $b^2 = 36$. This is exactly what we are looking for since $r \times g \times b = b^2$. Therefore, the product of the three numbers is 36.

For those who need to know what the actual numbers are, we can proceed and find the three numbers. We know $b^2 = 36$, so $b = 6$ since b is a positive number. Therefore, $g = 5 \times b = 5 \times 6 = 30$. And finally, $r \times g = b$ so $r \times (30) = 6$. Dividing by 30, we get $r = \frac{6}{30} = \frac{1}{5}$. We can verify the product $r \times g \times b = (\frac{1}{5}) \times (30) \times (6) = 6 \times 6 = 36$.

Solution 2

In this solution we will try to find the numbers by working with the factors of 180.

The product of the number on the card with a G and the number on the card with a B is 180 and the number on the card with a G is five times the number on the card with a B. The number 180 can be written as $2 \times 2 \times 3 \times 3 \times 5$. By playing with the factors, we can get the number on the card with a G is $5 \times 2 \times 3$ and the number on the card with a B is 2×3. That is, the number on the card with a G could be 30 and the number on the card with a B could be 6.

Now using the fact that the number on the card with an R times the number on the card with a G is equal to the number on the card with a B, we see that some number times 30 equals 6 and it follows that the number on the card with an R would be $6 \div 30 = \frac{1}{5}$.

The product of the three numbers is $\frac{1}{5} \times 30 \times 6 = 6 \times 6 = 36$.

This solution only works because the number on the card with a G and the number on the card with a B happen to be integers.

Problem of the Week

Problem C

Missed Out

A class of 30 students went on a field trip to a nature centre. Throughout the day, two different workshops were offered: “Know Your Birds” and “Animals Around Us”. Twelve of the students attended the workshop about birds. Seventeen of the students attended the “Animals Around Us” workshop. Five students attended both workshops. Some students did not attend any of the workshops.

How many students missed out by not attending any workshop during the field trip?


Themes Data Management, Number Sense

Problem of the Week
Problem C and Solution
Missed Out

Problem

A class of 30 students went on a field trip to a nature centre. Throughout the day, two different workshops were offered: "Know Your Birds" and "Animals Around Us". Twelve of the students attended the workshop about birds. Seventeen of the students attended the "Animals Around Us" workshop. Five students attended both workshops. Some students did not attend any of the workshops. How many students missed out by not attending any workshop during the field trip?

Solution

Since 5 students attended both workshops and these students are included in the 12 who attended the "Know Your Birds" workshop, then $12 - 5$ or 7 students attended only the "Know Your Birds" workshop. They did not attend the "Animals Around Us"' workshop.

Again, since 5 students attended both workshops and these students are included in the 17 who attended the "Animals Around Us" workshop, then $17 - 5$ or 12 students attended only the "Animals Around Us" workshop. They did not attend the "Know Your Birds" workshop as well.

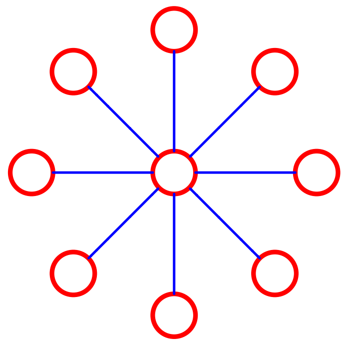
Students will be in exactly one of four possible groups: they attended both workshops, they attended the "Know Your Birds" workshop only, they attended the "Animals Around Us" workshop only, or they did not attend either workshop. The number of students in each possible group added together will sum to the total number of students in the class. Or we could subtract the known sizes of the groups from the total class size to determine the number of students who did not attend either workshop.

So, the number of students who did not attend either workshop is equal to the number of students in the class minus the number of students who attended both workshops minus the number of students attended only the "Know Your Birds" workshop minus the number of students who attended only the "Animals Around Us" workshop. Therefore, the number of students who did not attend either workshop is equal to $30 - 5 - 7 - 12 = 6$.

Therefore, 6 students missed out on the opportunity to learn from the workshop leaders.

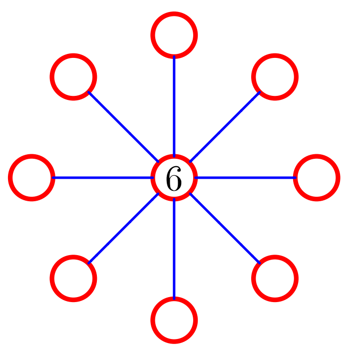
Problem of the Week
Problem C
Fill to 15

The game, "Fill to 15", is a two-player game. The game board consists of 9 circles as shown in the following diagram.



The players alternate turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once in any game. The object of the game is to be the first player to place a disc so that the sum of the 3 numbers along a line through the centre circle is exactly 15.

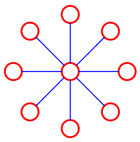
Alex and Blake play the game. Alex goes first. On her first move, Alex places a 6 in the centre circle. This is shown on the following diagram.



Then Blake places one of the eight remaining numbers in one of the empty circles on her first turn.

Show that no matter which numbers Blake plays, Alex can win the game on either her second or third turn.

Themes Computational Thinking, Number Sense



Problem of the Week

Problem C and Solution

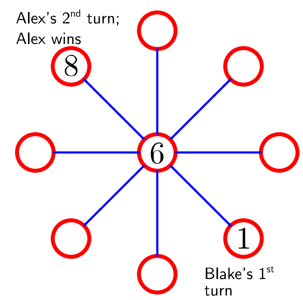
Fill to 15

Problem

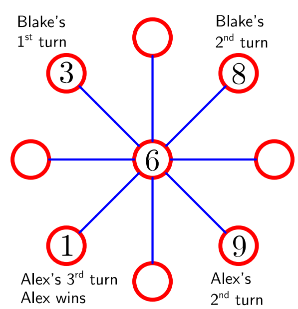
The game, "Fill to 15", is a two-player game. The game board consists of 9 circles as shown in the diagram above. The players alternate turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once in any game. The object of the game is to be the first player to place a disc so that the sum of the 3 numbers along a line through the centre circle is exactly 15. Alex and Blake play the game. Alex goes first. On her first move, Alex places a 6 in the centre circle. Then Blake places one of the eight remaining numbers in one of the empty circles on her first turn. Show that no matter which numbers Blake plays, Alex can win the game on either her second or third turn.

Solution

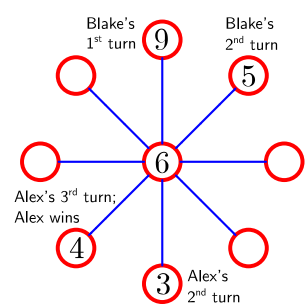
Since Alex played a 6 on her first turn, the other two discs in the line would need to add to 9 to make the total 15.



If, on her first turn, Blake plays one of the numbers 1, 2, 4, 5, 7 or 8, then there is an unused number that Alex can play on her second turn so that the sum of the line is 15. That is, if Blake plays a 1, then Alex will play an 8. (This is illustrated on the diagram to the right.) If Blake plays a 2, then Alex will play a 7. If Blake plays a 4, then Alex will play a 5. If Blake plays a 5, then Alex will play a 4. If Blake plays a 7, then Alex will play a 2. And if Blake plays an 8, then Alex will play a 1. In each of these 6 instances Alex can win on her second turn.



If, on her first turn, Blake places a 3 in any empty space, then the sum of the two discs in that line will be 9. Alex cannot win on her second turn since the only way to make the sum in that line 15 would be for her to play another 6. No number may be used more than once so this is not possible. However, if Alex completes the line by playing a 9 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7 and 8. Then, as we saw above, no matter what Blake plays on her second turn, there will be a number that Alex can place on that line so that the three numbers in the line add to 15. An example is illustrated in the diagram to the right.



Finally, if on her first turn, Blake places a 9 in any empty space, then the sum of the two discs on the line will already be 15 using just 2 discs. Alex cannot win on her second turn since playing any other disc in that line would make the sum greater than 15. However, if Alex completes the line by playing a 3 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7 and 8. Then, as we saw above, no matter what Blake plays next, there will be a number that Alex can place on that line so that the three numbers in the line add to 15. An example is illustrated in the diagram to the right.

Therefore, we have shown that no matter which numbers Blake plays, Alex can win the game on either her second or third turn.

Problem of the Week

Problem C

Location, Location, Location

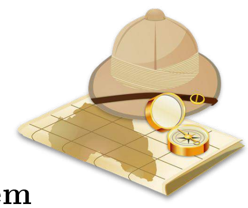
Starting with 2, we will place the integers as shown in the following chart.

	U	V	W	X	Y
Row 1			2	3	4
Row 2	7	6	5		
Row 3			8	9	10
Row 4	13	12	11		
Row 5			14	15	16
Row 6	19	18	17		

The pattern continues. Each new row lists the next three integers, in the direction opposite to the row above, and with the smallest of the three integers in column W.

Determine the exact location of the integer 2020. State the row number and the column letter (U, V, W, X, Y).

Themes Algebra, Number Sense



Problem of the Week

Problem C and Solution

Location, Location, Location

Problem

Starting with 2, we will place the integers as shown in the following chart.

	U	V	W	X	Y
Row 1			2	3	4
Row 2	7	6	5		
Row 3			8	9	10
Row 4	13	12	11		
Row 5			14	15	16
Row 6	19	18	17		

The pattern continues. Each new row lists the next three integers, in the direction opposite to the row above, and with the smallest of the three integers in column W.

Determine the exact location of the integer 2020. State the row number and the column letter (U, V, W, X, Y).

Solution

Observe some of the patterns in the chart. (There are many more patterns than the ones listed below.)

Each row contains a multiple of three in either column V or column X. Even multiples of three are in column V and odd multiples of three are in column X. To determine the row number, take the multiple of three and divide it by 3.

The outer numbers in column U or column Y have a remainder 1 when divided by 3. Numbers that are even and have a remainder 1 when divided by 3 are in column Y. Numbers that are odd and have a remainder 1 when divided by 3 are in column U. If the largest number in a row is even, it is in column Y. If the largest number in a row is odd, it is in column U.

Every number in column W has a remainder 2 when divided by 3.

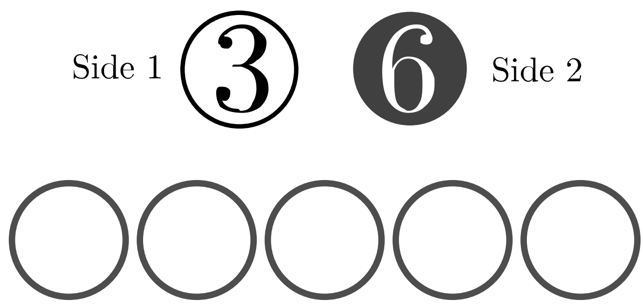
When 2020 is divided by 3, there is a quotient of 673 and a remainder 1. So 2020 is in column U or Y but since 2020 is even, it is in column Y. The 673^{rd} multiple of 3, which is 2019, is in row 673, in column X, to the left of 2020. In fact, row 673 will contain 2018 in column W; 2019, the 673^{rd} multiple of 3, in column X; and 2020 in column Y. Therefore, the number 2020 is located in row 673, column Y.

Problem of the Week

Problem C

To Create Numbers You Will Flip Over

A student has five identical disks. One side of each disk is white with a black number 3 stamped on it, and the other side of each disk is black with a white number 6 stamped on it.



Each of the five disks is placed on a different empty circle creating a five-digit number. How many different five-digit numbers can be formed such that the five-digit number is divisible by 9?

Did You Know?

If the sum of the digits of a number is divisible by 9, then the number is divisible by 9. If the sum of the digits of a number is not divisible by 9, then the number is not divisible by 9.

For example, the five-digit number 10 278 is divisible by 9 since the sum of the digits is $1 + 0 + 2 + 7 + 8 = 18$, which is divisible by 9. However, since the sum of the digits of the number 12 345 is 15 and 15 is not divisible by 9, then the number 12 345 is not divisible by 9.

③ ❻

Problem of the Week
Problem C and Solution
To Create Numbers You Will Flip Over

Problem

A student has five identical disks. One side of each disk is white with a black number 3 stamped on it, and the other side of each disk is black with a white number 6 stamped on it. Each of the five disks is placed on a different empty circle creating a five-digit number. How many different five-digit numbers can be formed such that the five-digit number is divisible by 9?

Solution

Solution 1

It is possible that some solvers listed every possible number and then tested to see which ones were divisible by 9. It turns out that there are only 32 possible numbers. After determining the digit sums, you would discover that 10 of the numbers are divisible by 9. No attempt will be made here to create the entire list.

Solution 2

In this solution we examine the possibilities by considering the number of each digit in the five-digit number.

1. The number contains five 3s and no 6s.

   The number contains five 3s and no 6s, so the digit sum is $5 \times 3 + 0 \times 6 = 15$. Since the digit sum 15 is not divisible by 9, no five-digit number containing this combination of digits is divisible by 9.

2. The number contains four 3s and one 6.

   The number contains four 3s and one 6, so the digit sum is $4 \times 3 + 1 \times 6 = 18$. Since the digit sum 18 is divisible by 9, the five-digit numbers containing this combination of digits are divisible by 9. These numbers are 33 336, 33 363, 33 633, 36 333, and 63 333. There are 5 such numbers.

3. The number contains three 3s and two 6s.

   The number contains three 3s and two 6s, so the digit sum is $3 \times 3 + 2 \times 6 = 21$. Since the digit sum 21 is not divisible by 9, no five-digit number containing this combination of digits is divisible by 9.

4. The number contains two 3s and three 6s.

   The number contains two 3s and three 6s, so the digit sum is $2 \times 3 + 3 \times 6 = 24$. Since the digit sum 24 is not divisible by 9, no five-digit number containing this combination of digits is divisible by 9.

5. The number contains one 3 and four 6s.

   The number contains one 3 and four 6s, so the digit sum is $1 \times 3 + 4 \times 6 = 27$. Since the digit sum 27 is divisible by 9, the five-digit numbers containing this combination of digits are divisible by 9. These numbers are 66 663, 66 636, 66 366, 63 666, and 36 666. There are 5 such numbers.

6. The number contains no 3s and five 6s.

   The number contains no 3s and five 6s, so the digit sum is $0 \times 3 + 5 \times 6 = 30$. Since the digit sum 30 is not divisible by 9, no five-digit number containing this combination of digits is divisible by 9.

Combining all of the above cases, there are $0 + 5 + 0 + 0 + 5 + 0 = 10$ five-digit numbers that can be formed from the given digits which are divisible by 9. The 10 numbers are 33 336, 33 363, 33 633, 36 333, 63 333, 66 663, 66 636, 66 366, 63 666, and 36 666.

Problem of the Week
Problem C
Accumulating Change

Canadian one-dollar coins are referred to as *loonies*. Canadian two-dollar coins are referred to as *toonies*.

Penny Saver has been saving quarters, loonies and toonies in her coin bank for a long time. No other types of coins are in her bank. One third of the coins in the bank are quarters and one fifth of the coins are loonies. There are 56 toonies in the bank.

Determine how much money Penny has saved in her coin bank.

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Accumulating Change

Problem
Penny Saver has been saving quarters, loonies and toonies in her coin bank for a long time. No other types of coins are in her bank. One third of the coins in the bank are quarters and one fifth of the coins are loonies. There are 56 toonies in the bank. Determine how much money Penny has saved in her coin bank.

Solution
Solution 1

One of the key sentences in the problem is "No other types of coins are in her bank." Using the fractions given, it will be possible to determine what fraction of the whole is made up by toonies.

$$\text{The fraction of toonies in the bank} = 1 - \frac{1}{3} - \frac{1}{5} = \frac{15}{15} - \frac{5}{15} - \frac{3}{15} = \frac{7}{15}.$$

We can now determine the total number of coins in the bank. Since 56 toonies are in the bank and $\frac{7}{15}$ of the coins are toonies,

$$\begin{array}{rlcl} & \frac{7}{15} \text{ of the coins in the bank} & = & 56 \text{ coins.} \\ \text{Dividing by 7,} & \frac{1}{15} \text{ of the coins in the bank} & = & 56 \div 7 = 8 \text{ coins.} \\ \text{Multiplying by 15,} & \frac{15}{15} \text{ of the coins in the bank} & = & 8 \times 15 = 120 \text{ coins.} \end{array}$$

There are 120 coins in the bank. We can now determine the number of quarters and loonies. The number of quarters $= \frac{1}{3} \times 120 = 40$ and the number of loonies $= \frac{1}{5} \times 120 = 24$.

To determine the amount of money in the bank, we multiply the value of a particular coin by the quantity of that coin and add the three values together.

$$\begin{array}{rcl} \text{Amount in the Bank} & = & \text{Value of Quarters} + \text{Value of Loonies} + \text{Value of Toonies} \\ & = & \$0.25 \times 40 \quad + \quad \$1.00 \times 24 \quad + \quad \$2 \times 56 \\ & = & \$10.00 \quad + \quad \$24.00 \quad + \quad \$112.00 \\ & = & \$146.00 \end{array}$$

Therefore, Penny has a total of \$146 in her bank.

A second solution, using equations, is found on the next page.

Solution 2

Let C represent the number of coins in the bank.

Then $\frac{1}{3}C$ is the number of quarters in the bank and $\frac{1}{5}C$ is the number of loonies in the bank.

It follows that $C - \frac{1}{3}C - \frac{1}{5}C = \frac{15}{15}C - \frac{5}{15}C - \frac{3}{15}C = \frac{7}{15}C$ is the number of toonies in the bank.

But there are 56 toonies in the bank, so

$$
\begin{aligned}
\frac{7}{15}C &= 56 \\
\frac{1}{15}C &= 8 && \text{(after dividing both sides by 7)} \\
C &= 120 && \text{(after multiplying both sides by 15)}
\end{aligned}
$$

There are 120 coins in the bank.

Then $\frac{1}{3} \times 120 = 40$ coins are quarters and $\frac{1}{5} \times 120 = 24$ coins are loonies.

To determine the amount of money in the bank, we multiply the value of a particular coin by the quantity of that coin and add the three values together.

$$
\begin{aligned}
\text{Amount in the Bank} &= \text{Value of Quarters} + \text{Value of Loonies} + \text{Value of Toonies} \\
&= \$0.25 \times 40 + \$1.00 \times 24 + \$2 \times 56 \\
&= \$10.00 + \$24.00 + \$112.00 \\
&= \$146.00
\end{aligned}
$$

Therefore, Penny has a total of \$146 in her bank.

Problem of the Week

Problem C

Sum of Everything

If you were to list the integers from 1 to 12, you would get the list

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$$

If you were to sum the digits of the integers in this list, you would get the sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + (1 + 0) + (1 + 1) + (1 + 2) = 51.$$

Below are the integers from 1 to 100. Can you find the sum of all of the digits of these numbers?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Themes Algebra, Number Sense

Problem of the Week

Problem C and Solution

Sum of Everything

Problem

If you were to list the integers from 1 to 12, you would get the list $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.
If you were to sum the digits of the integers in this list, you would get the sum
$1+2+3+4+5+6+7+8+9+(1+0)+(1+1)+(1+2) = 51$.

To the right are the integers from 1 to 100. Can you find the sum of all of the digits of these numbers?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Solution

(1) In the table above, each of the ten columns has a units digit that occurs ten times. So the sum of ALL of the units digits is

$$10(1)+10(2)+10(3)+10(4)+10(5)+10(6)+10(7)+10(8)+10(9)+10(0)$$
$$= 10(1+2+3+4+5+6+7+8+9+0)$$
$$= 10(45)$$
$$= 450$$

(2) Each of the ten columns has a tens digit from 0 to 9. So the sum of ALL of the tens digits is

$$10(0+1+2+3+4+5+6+7+8+9)$$
$$= 10(45)$$
$$= 450$$

(3) The number 100 is the only number with a hundreds digit. We need to add 1 to our final sum.

(4) Now we add our results from (1), (2), and (3) to obtain the required sum.

$$\begin{aligned} \text{Sum of digits} &= \text{Units digit sum} + \text{Tens digit sum} + \text{Hundreds Digit} \\ &= 450 + 450 + 1 \\ &= 901 \end{aligned}$$

Therefore, the sum of all of the digits of the numbers from 1 to 100 is 901.

Problem of the Week

Problem C

Faster!

Georgina enters a 12 km race. She wants to finish the race in one hour and twenty minutes. She starts off jogging at a speed of 7 km/h. After 30 minutes, she realizes that she needs to increase her speed to finish the race in her desired time. For the remaining time, what speed must she run at to finish the race in exactly one hour and twenty minutes?

Themes Geometry, Number Sense

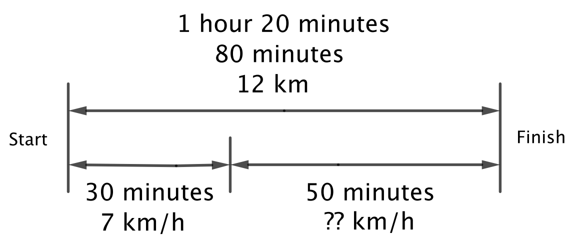
Problem of the Week
Problem C and Solution
Faster!

Problem

Georgina enters a 12 km race. She wants to finish the race in one hour and twenty minutes. She starts off jogging at a speed of 7 km/h. After 30 minutes, she realizes that she needs to increase her speed to finish the race in her desired time. For the remaining time, what speed must she run at to finish the race in exactly one hour and twenty minutes?

Solution

Representing the information in a diagram can be helpful when solving a problem like this.



The total trip is one hour and twenty minutes or 80 minutes. For the first 30 minutes, Georgina travels at a constant rate of 7 km/h. This means that in one hour (60 minutes) she would travel 7 km. Therefore, in half the time or 30 minutes she would travel half the distance or $7 \div 2 = 3.5$ km.

So Georgina must run $12 - 3.5 = 8.5$ km in $80 - 30 = 50$ minutes.

We need to determine the constant rate that Georgina needs to run at to accomplish this. Georgina needs to run 8.5 km in 50 minutes. By dividing by 5, Georgina needs to run $8.5 \div 5 = 1.7$ km in $50 \div 5 = 10$ minutes. Multiplying each term by 6 , Georgina must run $1.7 \times 6 = 10.2$ km in $10 \times 6 = 60$ minutes(1 hour.)

Therefore, Georgina must run the remaining distance at 10.2 km/h to accomplish her goal of finishing the 12 km race in one hour and twenty minutes.

Problem of the Week

Problem C

Count Down to Zero

Every year there is a countdown to the New Year in Problemville. The timer starts at 20 and counts down to 0.

The display for the digits on the timer is made up of seven segments that are either lit or unlit. When the digit 8 is displayed, all seven segments are lit.

When the digit 1 is displayed, only two segments are lit and five segments are unlit.

In changing from digit to digit, a segment can change from lit to unlit, from unlit to lit, or could remain unchanged. For example, in changing from 5 to 4, three of the segments that were lit stay lit, one segment that was unlit stays unlit, one segment that was unlit becomes lit, and two of the segments that were lit become unlit. Therefore, there is a total of three changes of state when the timer changes from 5 to 4.

In counting down from 20 to 0, how many changes of state are there? In other words, determine the number of times segments are turned from unlit to lit plus the number of times segments are turned from lit to unlit. (Note that, in changing from 10 to 9, the left digit is turned completely off.)

The ten digits are shown below for your reference.

Problem of the Week

Problem C and Solution

Count Down to Zero

Problem

Every year there is a countdown to the New Year in Problemville. The timer starts at 20 and counts down to 0. The display for the digits on the timer is made up of seven segments that are either lit or unlit. When the digit 8 is displayed, all seven segments are lit.When the digit 1 is displayed, only two segments are lit and five segments are unlit. In changing from digit to digit, a segment can change from lit to unlit, from unlit to lit, or could remain unchanged. For example, in changing from 5 to 4, three of the segments that were lit stay lit, one segment that was unlit stays unlit, one segment that was unlit becomes lit, and two of the segments that were lit become unlit. Therefore, there is a total of three changes of state when the timer changes from 5 to 4. In counting down from 20 to 0, how many changes of state are there? In other words, determine the number of times segments are turned from unlit to lit plus the number of times segments are turned from lit to unlit. (Note that, in changing from 10 to 9, the left digit is turned completely off.)

Solution

1. Consider the changes to the tens digit as a result of counting from 20 to 0.

   When the counter changes from 20 to 19, the tens digit, 2, changes to a 1. As a result of this change, four segments change from lit to unlit, one segment changes from unlit to lit, and the other two segments remain unchanged. There are a total of $4 + 1 = 5$ state changes.

   When the counter changes from 10 to 9, the tens digit, 1, turns off. As a result of this change, two segments become unlit, and the remaining five segments remain unchanged. There are 2 state changes to the tens digit when counting from 10 to 9.

   During the entire process of counting from 20 to 0, there are a total of $5 + 2 = 7$ changes in the state of the segments used for the tens digit.

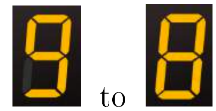
2. Consider the changes to the units digit as a result of counting from 20 to 0.

   The total number of changes in the units digit going from 20 to 10 is exactly the same as the number of changes in the units digit going from 10 to 0. We will count the number of changes going from 20 to 10 and double our result.

   In going from 20 to 19, the units digit changes from 0 to 9. One segment goes from lit to unlit, one segment goes from unit to lit, the remaining five segments remain unchanged. There is a total of 2 changes.

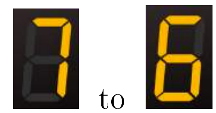
    to 

In going from 19 to 18, the units digit changes from 9 to 8. One segment goes from unlit to lit and the remaining six segments remain unchanged. There is 1 change.

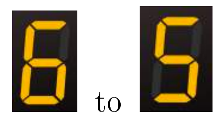


In going from 18 to 17, the units digit changes from 8 to 7. Four segments go from lit to unlit and the remaining three segments remain unchanged. There is a total of 4 changes.

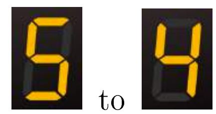
In going from 17 to 16, the units digit changes from 7 to 6. Four segments go from unlit to lit, one segment goes from lit to unlit, and the other two segments remain unchanged. There is a total of 5 changes.



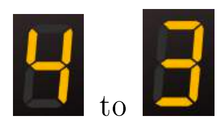
In going from 16 to 15, the units digit changes from 6 to 5. One segment goes from lit to unlit, and the other six segments remain unchanged. There is 1 change.



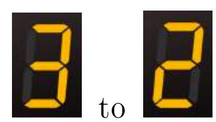
In going from 15 to 14, the units digit changes from 5 to 4. Two segments go from lit to unlit, one segment goes from unlit to lit and the other four segments remain unchanged. There is a total of 3 changes.



In going from 14 to 13, the units digit changes from 4 to 3. One segment goes from lit to unlit, two segments go from unlit to lit and the other four segments remain unchanged. There is a total of 3 changes.

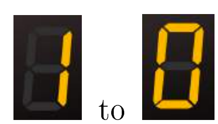


In going from 13 to 12, the units digit changes from 3 to 2. One segment goes from lit to unlit, one segment goes from unlit to lit and the other five segments remain unchanged. There is a total of 2 changes.



In going from 12 to 11, the units digit changes from 2 to 1. Four segments goes from lit to unlit, one segment goes from unlit to lit and the other two segments remain unchanged. There is a total of 5 changes.

In going from 11 to 10, the units digit changes from 1 to 0. Four segments goes from unlit to lit and the other three segments remain unchanged. There is a total of 4 changes.



So, in counting from 20 to 10 there is a total of $2+1+4+5+1+3+3+2+5+4 = 30$ changes to the units digit. Counting from 10 to 0 will produce the same number changes to the units digit that occurred in going from 20 to 10 so we must count another 30 changes.

In total, there are 7 changes in the tens digit and 60 changes in the units digit, for a total of 67 segment changes in counting down from 20 to 0.

Problem of the Week

Problem C

Ch-ch-changes

Bill made some purchases that totalled \$18.75 and paid for them with a twenty-dollar bill. The cash register has only quarters, dimes and nickels.

In how many different ways can the cashier make change?

"TOONIE" 2 dollar coin	"LOONIE" 1 dollar coin	QUARTER	NICKEL	DIME
200 cents	100 cents	25 cents	5 cents	10 cents

Remember, in this problem, we will only be using quarters, nickels and dimes.

Problem of the Week
Problem C and Solution
Ch-ch-changes

Problem

Bill made some purchases that totalled \$18.75 and paid for them with a twenty-dollar bill. The cash register has only quarters, dimes and nickels. In how many different ways can the cashier make change?

Solution

This is a good problem for applying a systematic approach.

The amount of change required is $\$20 - \$18.75 = \$1.25$ or 125 cents. In order to get to 125 cents, a maximum of 5 quarters are required. Once we determine the amount still required after the value of the quarters has been removed, we can determine the number of different combinations of dimes that can be given. For each of these possibilities, the remainder of the change will be nickels.

If 5 quarters are given as part of the change, the \$1.25 required as change is covered and no other coins are required. There is only 1 possibility for change in which 5 quarters are part of the change.

If 4 quarters are given as part of the change, \$0.25 is still required. There are 3 possibilities for dimes; either 0, 1 or 2 dimes. Therefore, there are 3 different coin combinations possible in which 4 quarters are part of the change.

If 3 quarters are given as part of the change, \$0.50 is still required. There are 6 possibilities for dimes; either 0, 1, 2, 3, 4, or 5 dimes. Therefore, there are 6 different coin combinations possible in which 3 quarters are part of the change.

If 2 quarters are given as part of the change, \$0.75 is still required. There are 8 possibilities for dimes; either 0, 1, 2, 3, 4, 5, 6, or 7 dimes. Therefore, there are 8 different coin combinations possible in which 2 quarters are part of the change.

If 1 quarter is given as part of the change, \$1.00 is still required. There are 11 possibilities for dimes; either 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 dimes. Therefore, there are 11 different coin combinations possible in which 1 quarter is part of the change.

If no quarters are given as part of the change, \$1.25 is still required. There are 13 possibilities for dimes; either 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 dimes. Therefore, there are 13 different coin combinations possible in which no quarters are part of the change.

The cashier can make the required change using $1 + 3 + 6 + 8 + 11 + 13 = 42$ different possible combinations of coins.

The solution is presented in chart form on the following page.

Number of Quarters	Value of Quarters (in cents)	Amount Remaining (in cents)	Number of Dimes	Value of Dimes (in cents)	Amount Remaining (in cents)	Number of Nickels Required
5	125	0	0	0	0	0
4	100	25	2	20	5	1
			1	10	15	3
			0	0	25	5
3	75	50	5	50	0	0
			4	40	10	2
			3	30	20	4
			2	20	30	6
			1	10	40	8
			0	0	50	10
2	50	75	7	70	5	1
			6	60	15	3
			5	50	25	5
			4	40	35	7
			3	30	45	9
			2	20	55	11
			1	10	65	13
			0	0	75	15
1	25	100	10	100	0	0
			9	90	10	2
			8	80	20	4
			7	70	30	6
			6	60	40	8
			5	50	50	10
			4	40	60	12
			3	30	70	14
			2	20	80	16
			1	10	90	18
			0	0	100	20
0	0	125	12	120	5	1
			11	110	15	3
			10	100	25	5
			9	90	35	7
			8	80	45	9
			7	70	55	11
			6	60	65	13
			5	50	75	15
			4	40	85	17
			3	30	95	19
			2	20	105	21
			1	10	115	23
			0	0	125	25

The cashier can make the required change using $1 + 3 + 6 + 8 + 11 + 13 = 42$ different possible combinations of coins.

Problem of the Week

Problem C

What's Not to Love?

At the start of the school year, students in Mr. Pi's class were asked the following question: "Do you love Math?" They were only allowed to answer "yes" or "no", and everyone had to answer. Of the 30 students in the class, 21 answered "yes" and 9 answered "no".

That day, with every student present, the probability of randomly selecting a student who answered the question "yes" was $\frac{21}{30} = \frac{7}{10}$ and the probability of randomly selecting a student who answered the question "no" was $\frac{9}{30} = \frac{3}{10}$.

However, on one particular morning later in the year, the following information was known about the class:

- at least one of the students who had answered "yes" was absent and at least one of the students who had answered "no" was absent;
- more than half of the class was present; and
- the probability of randomly selecting a student who had answered the question "yes" was $\frac{3}{4}$.

Is there enough information to determine how many students were absent that particular morning? If yes, how many students were absent? If no, explain why not.

Themes Data Management, Number Sense

Problem of the Week
Problem C and Solution
What's Not to Love?

Problem

At the start of the school year, students in Mr. Pi's class were asked the following question: "Do you love Math?" They were only allowed to answer "yes" or "no", and everyone had to answer. Of the 30 students in the class, 21 answered "yes" and 9 answered "no". So, with every student present, the probability of randomly selecting a student who answered the question "yes" was $\frac{21}{30} = \frac{7}{10}$ and the probability of randomly selecting a student who answered the question "no" was $\frac{9}{30} = \frac{3}{10}$.

However, on one particular morning later in the year, the following information was known about the class: at least one of the students who had answered "yes" was absent and at least one of the students who had answered "no" was absent; more than half of the class was present; and the probability of randomly selecting a student who had answered the question "yes" was $\frac{3}{4}$.

Is there enough information to determine how many students were absent that particular morning? If yes, how many students were absent? If no, explain why not.

Solution

Since at least one student from each of the two groups was absent, there were at least 2 students absent and at most 28 students present. Also, the maximum number of students who said "yes" would be $21 - 1 = 20$ and the maximum number of students who said "no" would be $9 - 1 = 8$. More than half the class was present so at least 16 students were present.

Since the probability of randomly selecting a student who answered "yes" was $\frac{3}{4}$, then the probability of randomly selecting a student who answered "no" was $\frac{1}{4}$.

We are looking for any number from 16 to 28 which is divisible by 4, so that when we find $\frac{3}{4}$ and $\frac{1}{4}$ of this number our result is a whole number. The numbers from 16 to 28 that are divisible by 4 are 16, 20, 24 and 28.

The following chart shows the results which are possible using the given information. There are 3 valid solutions that satisfy the given information. Therefore, there is not enough given information to determine the number of students who were absent that particular morning. The last solution in the chart in not valid. If the number present was 28, then 21 of those present would have answered the question "yes". But at least one student who answered "yes" was absent so the maximum number of students who answered "yes" would have been 20.

Number Present	Number Absent	Number who said "yes"	Number who said "no"	Valid / Not Valid
16	$30 - 16 = 14$	$\frac{3}{4} \times 16 = 12$	$\frac{1}{4} \times 16 = 4$	Valid
20	$30 - 20 = 10$	$\frac{3}{4} \times 20 = 15$	$\frac{1}{4} \times 20 = 5$	Valid
24	$30 - 24 = 6$	$\frac{3}{4} \times 24 = 18$	$\frac{1}{4} \times 24 = 6$	Valid
28	$30 - 28 = 2$	$\frac{3}{4} \times 28 = 21$	$\frac{1}{4} \times 28 = 7$	Not Valid

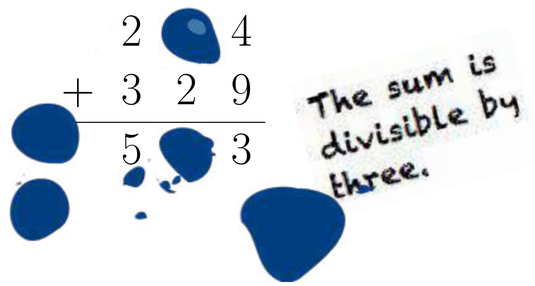
To Think About: Is there another piece of information that Mr. Pi could have provided so that two of the three valid answers could be eliminated leaving only one valid answer?

Problem of the Week

Problem C

What a Mess!

Jing was looking through an old math notebook and found the following mess on one of the pages:



The middle digit of the top and bottom numbers were both smudged and unreadable. Beside the question there was a handwritten note which read: "The sum is divisible by three."

If the missing middle digit in the top number is A and the missing middle digit in the bottom number is B, determine all possible values for A and B.

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
What a Mess!

Problem

Jing was looking through an old math notebook and found the following question on one of the pages. The middle digit of the top and bottom numbers were both smudged and unreadable. Beside the question there was a handwritten note which read: "The sum is divisible by three." If the missing middle digit in the top number is A and the missing middle digit in the bottom number is B, determine all possible values for A and B.

$$\begin{array}{cccc} & 2 & A & 4 \\ + & 3 & 2 & 9 \\ \hline & 5 & B & 3 \end{array}$$

Solution

Solution 1

First we find the possible values of B. For a number to be divisible by 3, the sum of its digits must be divisible by 3. So $5 + B + 3$ must be divisible by 3. The only possible values for B are thus 1, 4, or 7.

We know that $2A4 + 329 = 5B3$ so $2A4 = 5B3 - 329$.

We can try each of the possible values for B in the equation $2A4 = 5B3 - 329$ to find values of A that make the equation true.

1. If $B = 1$, then $513 - 329 = 184$, which cannot equal $2A4$. So when $B = 1$ there is no A to satisfy the problem.
2. If $B = 4$, then $543 - 329 = 214$, which does equal $2A4$ when $A = 1$. So for $A = 1$ and $B = 4$ there is a valid solution.
3. If $B = 7$, then $573 - 329 = 244$, which does equal $2A4$ when $A = 4$. So for $A = 4$ and $B = 7$ there is a valid solution.

Therefore, when $A = 1$ and $B = 4$ or when $A = 4$ and $B = 7$, the given problem has a valid solution.

Solution 2

$$\begin{array}{cccc} & 2 & A & 4 \\ + & 3 & 2 & 9 \\ \hline & 5 & B & 3 \end{array}$$

When the digits in the unit's column are added together, there is one carried to the ten's column. When the digits in the hundred's column are added together we get $2 + 3 = 5$ so there is no carry from the ten's column. Therefore, when the ten's column is added we get $1 + A + 2 = B$ or $A + 3 = B$.

We can now look at all possible values for A that produce a single digit value for B in the number $5B3$. We can then determine whether or not $5B3$ is divisible by 3.

The following table summarizes the results.

A	$B = A + 3$	$5B3$	Divisible by 3 (*yes*/*no*)?
0	3	533	*no*
1	4	543	*yes*
2	5	553	*no*
3	6	563	*no*
4	7	573	*yes*
5	8	583	*no*
6	9	593	*no*

Therefore, when $A = 1$ and $B = 4$ or when $A = 4$ and $B = 7$, the given problem has a valid solution.

Problem of the Week

Problem C

We Took the Cookies

There is a cookie jar that contains a certain number of cookies. Three friends divide the cookies in the following way. First, Harold takes all of the cookies and places them into three equal piles with none left over. He then keeps one of the piles and puts the other two piles back into the jar. Next, Lucie takes the remaining cookies in the jar and places them into three equal piles with none left over. She then keeps one of the piles and puts the other two piles back into the jar. Finally, Livio takes the remaining cookies in the jar and places them into three equal piles, but there is one cookie left over. He then keeps the leftover cookie and one of the piles and puts the other two piles back into the jar.

If there are now 10 cookies in the jar, how many cookies were originally in the cookie jar?

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
We Took the Cookies

Problem
There is a cookie jar that contains a certain number of cookies. Three friends divide the cookies in the following way. First, Harold takes all of the cookies and places them into three equal piles with none left over. He then keeps one of the piles and puts the other two piles back into the jar. Next, Lucie takes the remaining cookies in the jar and places them into three equal piles with none left over. She then keeps one of the piles and puts the other two piles back into the jar. Finally, Livio takes the remaining cookies in the jar and places them into three equal piles, but there is one cookie left over. He then keeps the leftover cookie and one of the piles and puts the other two piles back into the jar. If there are now 10 cookies in the jar, how many cookies were originally in the cookie jar?

Solution

We will present two different solutions. Solution 1 works backwards through the problem. Solution 2 is an algebraic solution.

Solution 1

The last 10 cookies in the cookie jar are also the remaining two of Livio's three piles. Therefore, each pile he made had $10 \div 2 = 5$ cookies. Therefore, there were $5 \times 3 = 15$ cookies in the three piles he made plus 1 more cookie that he kept, for a total of 16 cookies. Therefore, there were 16 cookies in the cookie jar when Livio started dividing the cookies.

These 16 cookies were two of the three piles that Lucie made. Therefore, each pile that she made had $16 \div 2 = 8$ cookies. Therefore, there were $8 \times 3 = 24$ cookies in the three piles that she made. Therefore, there were 24 cookies in the cookie jar when Lucie started dividing the cookies.

These 24 cookies were two of the three piles that Harold made. Therefore, each pile that he made had $24 \div 2 = 12$ cookies. Therefore, there were $12 \times 3 = 36$ cookies in the three piles that he made. Therefore, there were 36 cookies in the cookie jar when Harold started dividing the cookies.

Therefore, there were originally 36 cookies in the cookie jar.

Solution 2

Let the initial number of cookies in the cookie jar be C.

Harold has $\frac{1}{3}C$ cookies in the pile he keeps. Therefore, $\frac{2}{3}C$ cookies are left for Lucie.

Lucie keeps $\frac{1}{3}$ of $\frac{2}{3}C$ cookies. Therefore, $\frac{2}{3}$ of $\frac{2}{3}C$ cookies are left for Livio. That is, $\frac{2}{3} \times \frac{2}{3}C = \frac{4}{9}C$ cookies are left for Livio.

For Livio, the pile he keeps is $\frac{1}{3}$ of one less than what is left. That is $\frac{1}{3} \times \left(\frac{4}{9}C - 1\right)$, and so the remaining number of cookies that he puts back into the cookie jar is equal to $\frac{2}{3} \times \left(\frac{4}{9}C - 1\right)$.

This is also equal to 10. That is,

$$\frac{2}{3} \times \left(\frac{4}{9}C - 1\right) = 10$$

Dividing both sides by $\frac{2}{3}$,

$$\frac{\frac{2}{3} \times \left(\frac{4}{9}C - 1\right)}{\frac{2}{3}} = \frac{10}{\frac{2}{3}}$$

Since $10 \div \frac{2}{3} = 10 \times \frac{3}{2} = \frac{30}{2} = 15$,

$$\frac{4}{9}C - 1 = 15$$

Therefore,

$$\frac{4}{9}C = 16$$

Dividing both sides by $\frac{4}{9}$,

$$\frac{\frac{4}{9}C}{\frac{4}{9}} = \frac{16}{\frac{4}{9}}$$

Since $16 \div \frac{4}{9} = 16 \times \frac{9}{4} = 36$,

$$C = 36$$

Therefore, there were originally 36 cookies in the cookie jar.

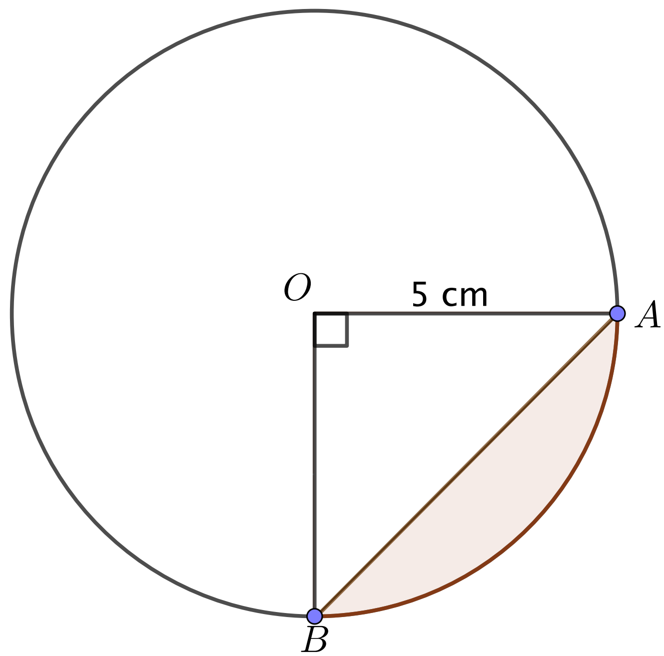
Geometry (G)

TAKE ME TO THE COVER

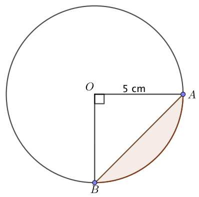
Problem of the Week
Problem C
Circular Segment

In the diagram, $\angle BOA = 90^\circ$ and the radius of the circle, OA, is 5 cm.

Determine the area of the shaded region correct to one decimal place.



Theme Geometry



Problem of the Week

Problem C and Solution

Circular Segment

Problem

In the diagram above, $\angle BOA = 90^\circ$ and the radius of the circle, OA, is 5 cm. Determine the area of the shaded region correct to one decimal place.

Solution

Together, $\triangle BOA$ and the shaded region cover $\frac{1}{4}$ of the area of the circle.

To find the area of the shaded region, we need to find the area of the triangle and subtract it from one-quarter of the area of the circle.

Since $\angle BOA = 90^\circ$, OA and OB meet at 90°. We can then use OA as the base and OB as the height in the formula for the area of a triangle to find the area of $\triangle BOA$.

$$\begin{aligned} \text{Area } \triangle BOA &= \frac{\text{base} \times \text{height}}{2} \\ &= \frac{OA \times OB}{2} \\ &= \frac{5 \times 5}{2} \qquad \text{(}OA \text{ and } OB \text{ are radii of the circle.)} \\ &= \frac{25}{2} \\ &= 12.5 \text{ cm}^2 \end{aligned}$$

To find the area of the quarter circle, we will use the formula for the area of a circle, $A = \pi r^2$, and then divide the result by 4.

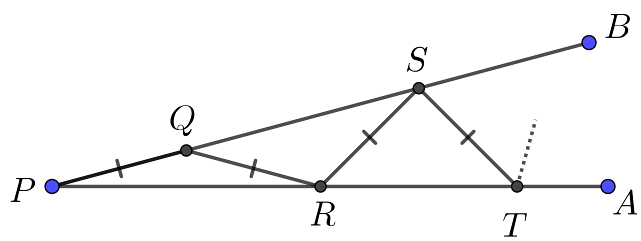
$$\begin{aligned} \text{Area of the Quarter Circle} &= \pi \times r^2 \div 4 \\ &= \pi \times 5 \times 5 \div 4 \\ &= \pi \times 25 \div 4 \\ &= (6.25 \times \pi) \text{ cm}^2 \end{aligned}$$

We can now determine the shaded area.

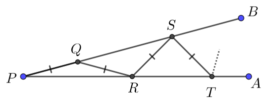
$$\begin{aligned} \text{Shaded Area} &= \text{Area of the Quarter Circle} - \text{Area } \triangle BOA \\ &= 6.25 \times \pi - 12.5 \\ &\doteq 7.1 \text{ cm}^2 \end{aligned}$$

Therefore, correct to one decimal place, the shaded area is 7.1 cm^2.

Problem of the Week
Problem C
Arm to Arm



In the diagram, $\angle APB = 12^\circ$. Points $Q, R, S, T, \ldots$ alternate from one arm of the angle to the other, each point located farther away from P than the point before and $PQ = QR = RS = ST = \ldots$. Eventually, one of the isosceles triangles will be an equilateral triangle. How many isosceles triangles will be formed before the equilateral triangle is formed?



Problem of the Week

Problem C and Solution

Arm to Arm

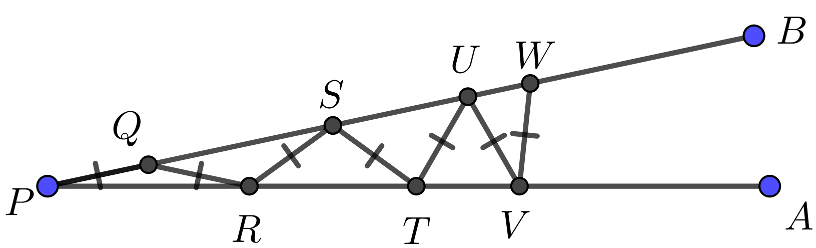
Problem

In the diagram, $\angle APB = 12^\circ$. Points $Q, R, S, T, \ldots$ alternate from one arm of the angle to the other, each point located farther away from P than the point before and $PQ = QR = RS = ST = \ldots$. Eventually, one of the isosceles triangles will be an equilateral triangle. How many isosceles triangles will be formed before the equilateral triangle is formed?

Solution

Throughout this solution there will be references to the exterior angle theorem (EAT). The theorem states: "An exterior angle of a triangle is equal to the sum of the opposite interior angles inside the triangle." For example, $\angle SQR$ is exterior to $\triangle QPR$ and $\angle SQR = \angle QPR + \angle QRP$.

Extend the diagram so a few more triangles are created such that $PQ = QR = RS = ST = TU = UV = VW$. In the solution we will show that there will be 4 isosceles triangles before the equilateral triangle is formed.



In $\triangle QPR$, $\angle QPR = 12^\circ$ and $PQ = QR$.
Therefore, $\triangle QPR$ is isosceles and $\angle QRP = \angle QPR = 12^\circ$. $\angle SQR$ is exterior to $\triangle QPR$ so, by EAT, $\angle SQR = \angle QPR + \angle QRP = 12^\circ + 12^\circ = 24^\circ$.

In $\triangle RQS$, $\angle SQR = 24^\circ$ and $QR = RS$.
Therefore, $\triangle RQS$ is isosceles and $\angle RSQ = \angle SQR = 24^\circ$. $\angle SRT$ is exterior to $\triangle PRS$ so, by EAT, $\angle SRT = \angle SPR + \angle PSR = 12^\circ + 24^\circ = 36^\circ$.

In $\triangle SRT$, $\angle SRT = 36^\circ$ and $SR = ST$.
Therefore, $\triangle SRT$ is isosceles and $\angle STR = \angle SRT = 36^\circ$. $\angle UST$ is exterior to $\triangle PST$ so, by EAT, $\angle UST = \angle SPT + \angle STP = 12^\circ + 36^\circ = 48^\circ$.

In $\triangle TSU$, $\angle UST = 48^\circ$ and $ST = TU$.
Therefore, $\triangle TSU$ is isosceles and $\angle TUS = \angle UST = 48^\circ$. $\angle UTV$ is exterior to $\triangle PUT$ so, by EAT, $\angle UTV = \angle UPT + \angle PUT = 12^\circ + 48^\circ = 60^\circ$.

In $\triangle UTV$, $\angle UTV = 60^\circ$ and $TU = UV$.
Therefore, $\triangle UTV$ is isosceles and $\angle UVT = \angle UTV = 60^\circ$. Since $\angle UVT = \angle UTV = 60^\circ$, the sum of these angles is 120° and the remaining angle $\angle TUV = 180^\circ - 120^\circ = 60^\circ$. Since all three angles equal 60°, the triangle is an equilateral triangle.

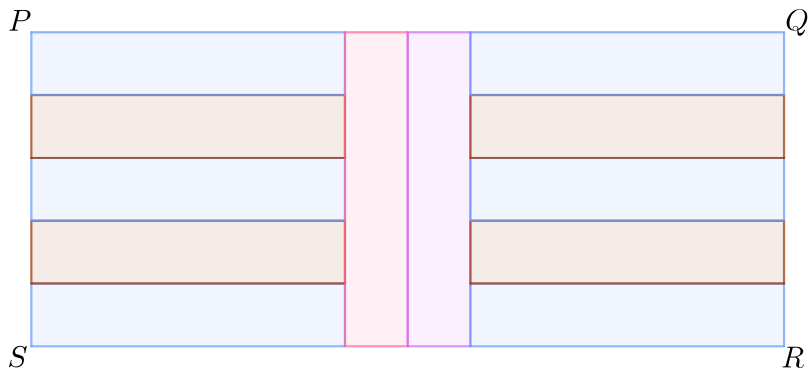
Since $\triangle UTV$ is the fifth triangle, there are 4 isosceles triangles before the equilateral triangle is formed.

EXTENSION: In our problem, we started with $\angle APB = 12^\circ$ and we eventually generated an equilateral triangle. What other values for $\angle APB$ will eventually generate an equilateral triangle?

Problem of the Week

Problem C

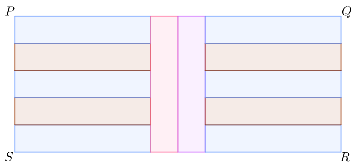
A Perfect Dozen



Twelve identical rectangles are arranged as shown in the diagram to form a large rectangle $PQRS$.

If the area of rectangle $PQRS$ is 540 cm^2, determine the dimensions of the smaller rectangles.

Themes Algebra, Geometry



Problem of the Week
Problem C and Solution
A Perfect Dozen

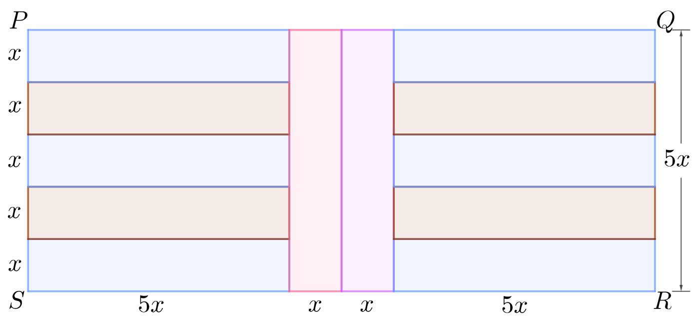
Problem

Twelve identical rectangles are arranged as shown in the diagram above to form a large rectangle $PQRS$. If the area of rectangle $PQRS$ is 540 cm^2, determine the dimensions of the smaller rectangles.

Solution

Solution 1

Let x be the width of one of the smaller identical rectangles, in cm. Five of the smaller rectangles are stacked on top of each other creating PS, so $PS = x + x + x + x + x = 5x$. Since $PQRS$ is a rectangle, $PS = QR = 5x$. But $5x$ then also is the length of a smaller rectangle. Therefore, a smaller rectangle is $5x$ cm by x cm. This information is all marked on the following diagram.



The area of rectangle $PQRS$ is the same as 12 times the area of one of the smaller rectangles.

$$\begin{aligned} \text{Area } PQRS &= 12 \times \text{Area of one smaller rectangle} \\ 540 &= 12 \times 5x \times x \\ 540 &= 60 \times x^2 \end{aligned}$$

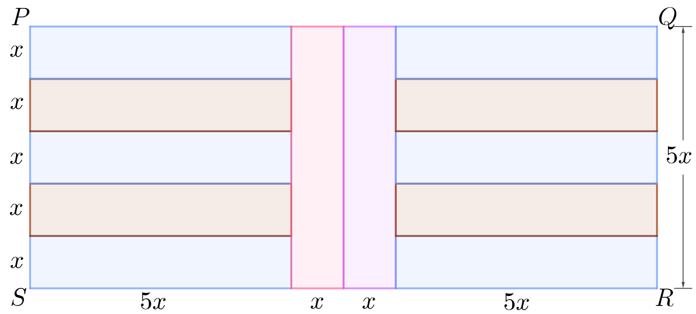
Dividing both sides by 60, we obtain $x^2 = 9$ and $x = 3$ follows. ($x > 0$ since x is the width of a smaller rectangle.)

The width of a smaller rectangle is $x = 3$ cm and the length of a smaller rectangle is $5x = 5(3) = 15$ cm.

Therefore, the smaller rectangles are each 15 cm long and 3 cm wide.

Solution 2

Let x be the width of one of the smaller identical rectangles, in cm. Five rectangles are stacked on top of each other creating PS, so $PS = x + x + x + x + x = 5x$.



Also $PQRS$ is a rectangle so $QR = PS = 5x$ and $5x$ is also the length of a smaller rectangle.

Now SR is made up of the lengths of two of the smaller rectangles plus the widths of two of the smaller rectangles. Therefore, $SR = 5x + 5x + x + x = 12x$ and rectangle $PQRS$ is $12x$ cm long and $5x$ cm wide.

To find the area of $PQRS$ we multiply the length SR by the width QR.

$$\begin{aligned} \text{Area } PQRS &= SR \times QR \\ 540 &= (12x) \times (5x) \\ 540 &= 12 \times 5 \times x \times x \\ 540 &= 60 \times x^2 \end{aligned}$$

Dividing by 60, we obtain $x^2 = 9$ and $x = 3$ follows. ($x > 0$ since x is the width of a smaller rectangle.)

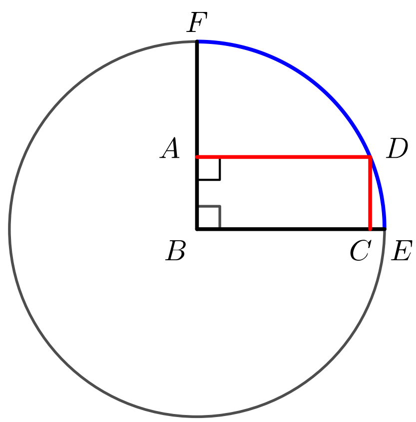
The width of a smaller rectangle is $x = 3$ cm and the length of a smaller rectangle is $5x = 5(3) = 15$ cm.

Therefore, the smaller rectangles are each 15 cm long and 3 cm wide.

Problem of the Week
Problem C
A Missing Length

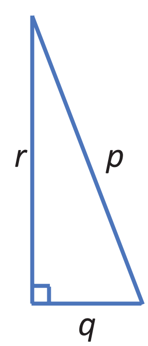
A circle with center B and radius 13 cm has three distinct points, F, D and E, on its circumference so that $BF \perp BE$ and D is on the minor arc FE. Point A is on BF so that $DA \perp BF$. The point C is on BE so that $ABCD$ is a rectangle and the distance from C to E is 1 cm.

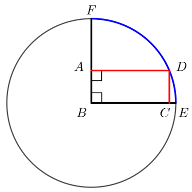
Determine the distance from A to F.



The *Pythagorean Theorem* states, "In a right triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides".

In the following right triangle, $p^2 = r^2 + q^2$.



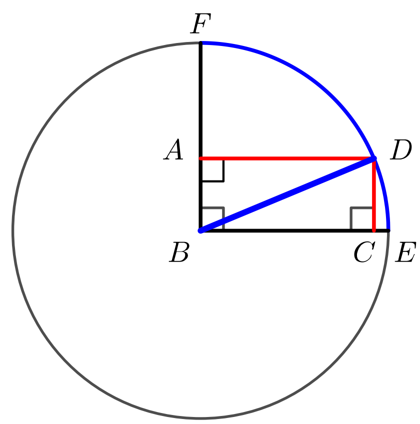


Problem of the Week
Problem C and Solution
A Missing Length

Problem

A circle with center B and radius 13 cm has three distinct points, F, D and E, on its circumference so that $BF \perp BE$ and D is on the minor arc FE. Point A is on BF so that $DA \perp BF$. The point C is on BE so that $ABCD$ is a rectangle and the distance from C to E is 1 cm. Determine the distance from A to F.

Solution



Construct radius BD.

Since the radius of the circle is 13 cm, $BF = BD = BE = 13$ cm.

Then $BC = BE - CE = 13 - 1 = 12$ cm.

Since $ABCD$ is a rectangle, $\angle BCD = 90^\circ$.

Using the Pythagorean Theorem in right $\triangle BCD$,

$$\begin{aligned} DC^2 &= DB^2 - BC^2 \\ &= 13^2 - 12^2 \\ &= 169 - 144 \\ &= 25 \\ DC &= 5 \text{ cm (since } DC > 0) \end{aligned}$$

Since $ABCD$ is a rectangle, $AB = DC = 5$ cm.

Then $AF = BF - AB = 13 - 5 = 8$ cm.

Therefore, the length of AF is 8 cm.

Problem of the Week

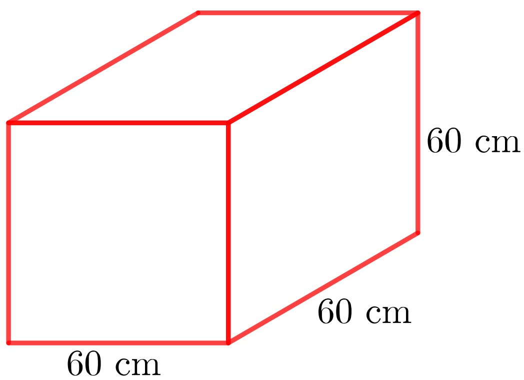
Problem C

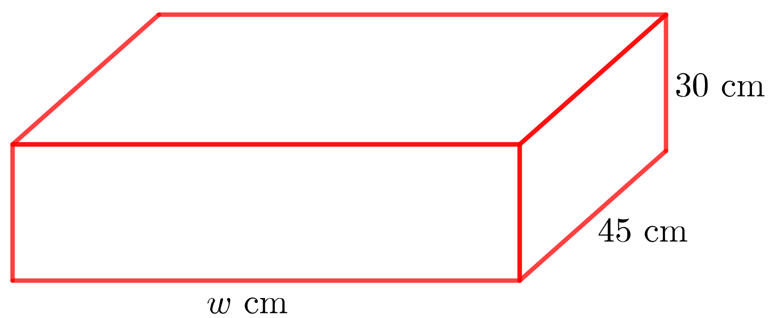
Can You Build It?

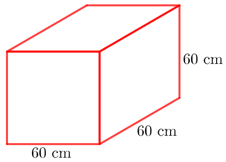
Two students are building storage boxes in woodworking class. The first student builds a box in the form of a cube with each edge measuring 60 cm. The second student builds a box in the shape of a rectangular prism with width w cm, height 30 cm and depth 45 cm.

Surprisingly, each student's box has the same total surface area.

Determine which student's box has the greatest volume. How much greater is the volume?



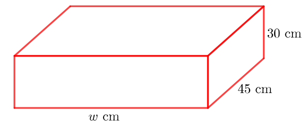




Problem of the Week

Problem C and Solution

Can You Build It?



Problem

Two students are building storage boxes in woodworking class. The first student builds a box in the form of a cube with each edge measuring 60 cm. The second student builds a box in the shape of a rectangular prism with width w cm, height 30 cm and depth 45 cm. Surprisingly, each student's box has the same total surface area. Determine which student's box has the greatest volume. How much greater is the volume?

Solution

To find the total surface area of a cube, determine the area of one side and multiply by 6. Therefore, the total surface area of the cube-shaped box is $6 \times 60 \times 60 = 21\,600$ cm^2.

To find the total surface area of the rectangular prism, determine the sum of the areas of the six sides. The front side and the back side have equal areas. The top and the bottom have equal areas. Each of the remaining two sides have equal area. Therefore, the total surface area of the rectangular prism box is

$$\begin{aligned} & 2 \times \text{area of front} + 2 \times \text{area of top} + 2 \times \text{area of side} \\ = \; & 2 \times 30 \times w + 2 \times 45 \times w + 2 \times 30 \times 45 \\ = \; & 60w + 90w + 2700 \\ = \; & 150w + 2700 \end{aligned}$$

But the total surface area of the cube-shaped box is the same as the total surface area of the rectangular prism. So,

$$\begin{aligned} 21\,600 &= 150w + 2700 \\ 21\,600 - 2700 &= 150w + 2700 - 2700 \\ 18\,900 &= 150w \\ \frac{18\,900}{150} &= \frac{150w}{150} \\ 126 &= w \end{aligned}$$

Therefore, the width of the rectangular prism is 126 cm.

To find the volume of a cube, "cube" the edge length. So, the volume of the cube is $60 \times 60 \times 60 = 60^3 = 216\,000$ cm^3.

To find the volume of the rectangular prism, multiply the three different edge lengths. So, the volume of the rectangular prism is $126 \times 45 \times 30 = 170\,100$ cm^3.

Therefore, the student who is building the cube has the box with the greater volume. The volume of the cube is greater by $216\,000 - 170\,100 = 45\,900$ cm^3.

Problem of the Week

Problem C

Faster!

Georgina enters a 12 km race. She wants to finish the race in one hour and twenty minutes. She starts off jogging at a speed of 7 km/h. After 30 minutes, she realizes that she needs to increase her speed to finish the race in her desired time. For the remaining time, what speed must she run at to finish the race in exactly one hour and twenty minutes?

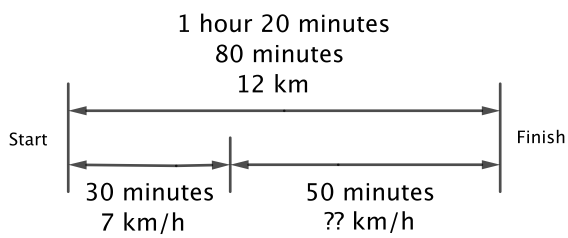
Problem of the Week
Problem C and Solution
Faster!

Problem

Georgina enters a 12 km race. She wants to finish the race in one hour and twenty minutes. She starts off jogging at a speed of 7 km/h. After 30 minutes, she realizes that she needs to increase her speed to finish the race in her desired time. For the remaining time, what speed must she run at to finish the race in exactly one hour and twenty minutes?

Solution

Representing the information in a diagram can be helpful when solving a problem like this.



The total trip is one hour and twenty minutes or 80 minutes. For the first 30 minutes, Georgina travels at a constant rate of 7 km/h. This means that in one hour (60 minutes) she would travel 7 km. Therefore, in half the time or 30 minutes she would travel half the distance or $7 \div 2 = 3.5$ km.

So Georgina must run $12 - 3.5 = 8.5$ km in $80 - 30 = 50$ minutes.

We need to determine the constant rate that Georgina needs to run at to accomplish this. Georgina needs to run 8.5 km in 50 minutes. By dividing by 5, Georgina needs to run $8.5 \div 5 = 1.7$ km in $50 \div 5 = 10$ minutes. Multiplying each term by 6 , Georgina must run $1.7 \times 6 = 10.2$ km in $10 \times 6 = 60$ minutes(1 hour.)

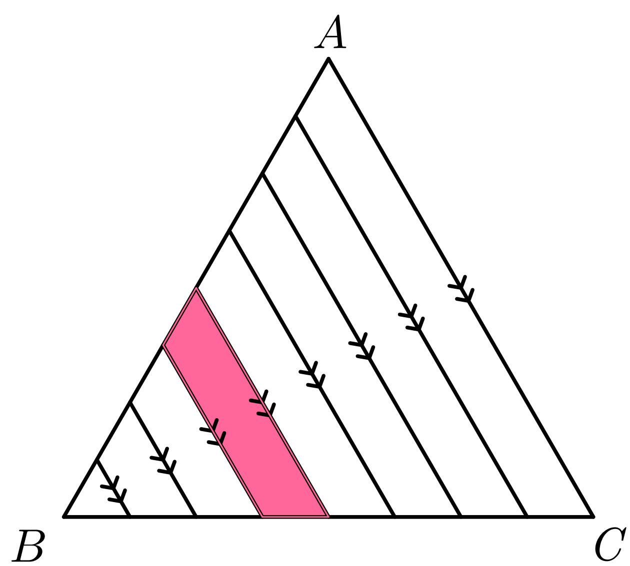
Therefore, Georgina must run the remaining distance at 10.2 km/h to accomplish her goal of finishing the 12 km race in one hour and twenty minutes.

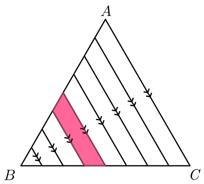
Problem of the Week
Problem C
Fence Me In

A triangular plot of land, labeled ABC in the diagram below, has equal side lengths. Each side is 120 m. The larger plot of land is divided into 8 smaller plots of land as follows:

- Sides BA and BC are each divided into 8 segments of equal length.
- Each point of division on BA is connected to its corresponding point of division on BC, creating 7 line segments.
- Each of the 7 line segments is parallel to the third side of the triangle, AC, as shown.

Eight plots of land are created by this process. The shaded section is to be surrounded by fence. How much fencing is required?

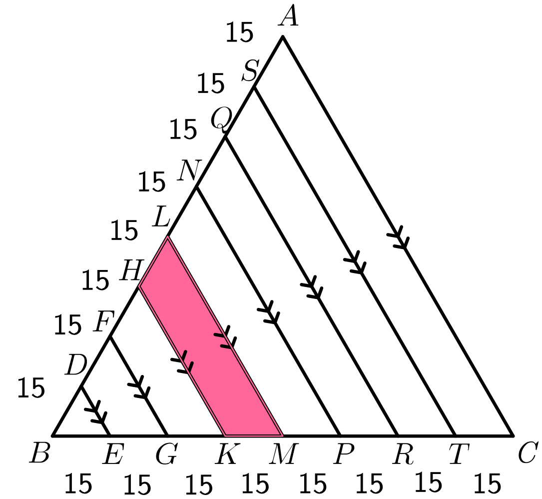




Problem of the Week
Problem C and Solution
Fence Me In

Problem

A triangular plot of land, labeled ABC in the diagram above, has equal side lengths. Each side is 120 m. The larger plot of land is divided into 8 smaller plots of land as follows: sides BA and BC are each divided into 8 segments of equal length, and each point of division on BA is connected to its corresponding point of division on BC, creating 7 line segments, each of which is parallel to the third side of the triangle, BC. This process creates eight plots of land. The shaded section is to be surrounded by fence. How much fencing is required?



Solution

Solution 1

The angles in an equilateral triangle are each 60°. Therefore

$$\angle BAC = \angle BCA = \angle ABC = 60^\circ.$$

Label the points of division as shown on the diagram above.

Since each of the sides BA and BC are sides of an equilateral triangle, $BA = BC = 120$ m. Each of these sides is divided into 8 segments of equal length. It follows that

$$BD = DF = FH = HL = LN = NQ = QS = SA = 120 \div 8 = 15 \text{ m and}$$

$$BE = EG = GK = KM = MP = PR = RT = TC = 120 \div 8 = 15 \text{ m.}$$

Since $HK \parallel LM \parallel AC$, $\angle BHK = \angle BLM = \angle BAC = 60^\circ$ and $\angle BKH = \angle BML = \angle BCA = 60^\circ$.

In $\triangle BHK$, $\angle BHK = \angle BKH = \angle HBK = 60^\circ$ and it follows that $\triangle BHK$ is equilateral. But $BH = BD + DF + FH = 15 + 15 + 15 = 45$, so $HK = 45$ m.

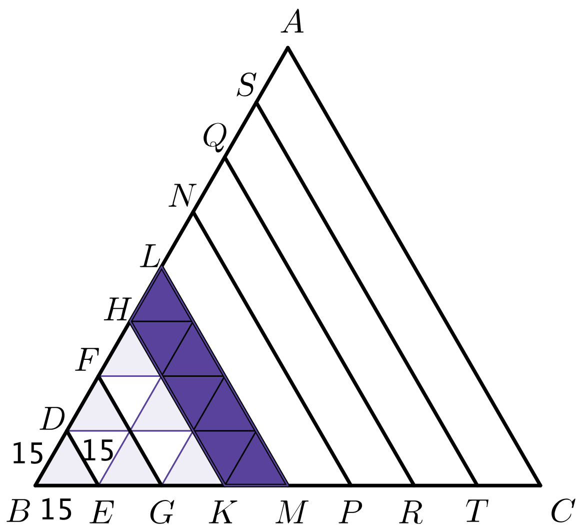
Similarly, in $\triangle BLM$, $\angle BLM = \angle BML = \angle LBM = 60^\circ$ and it follows that $\triangle BLM$ is equilateral. But $BL = BD + DF + FH + HL = 15 + 15 + 15 + 15 = 60$, so $LM = 60$ m.

The perimeter of the shaded region is $HL + LM + KM + HK = 15 + 60 + 15 + 45 = 135$ m. It follows that 135 m of fencing is required to enclose this shaded area.

Solution 2

Label the points of division as shown on the diagram below.

Observe that the large equilateral triangle can be tiled with copies of the smaller equilateral triangle with side length 15 m. A complete justification of this is not provided here but you may wish to verify this for yourself.



Three of the smaller equilateral triangles cover the entire area occupied by quadrilateral $DEGF$.

Five of the smaller equilateral triangles cover the entire area occupied by quadrilateral $FGKH$.

Seven of the smaller equilateral triangles cover the entire area occupied by quadrilateral $HKML$.

If we were to continue, $\triangle ABC$ would contain

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64$$

of the smaller equilateral triangles.

In quadrilateral $HKML$, the smaller side, HK, contains the bases of three of the smaller equilateral triangles and therefore is 45 m long. The larger side, LM, contains the bases of four of the smaller equilateral triangles and therefore is 60 m long.

The perimeter is the sum of the lengths of the four sides of quadrilateral $HKML$. Therefore, the perimeter of the shaded region is $15 + 15 + 45 + 60 = 135$ m. It follows that 135 m of fencing is required to enclose this shaded area.

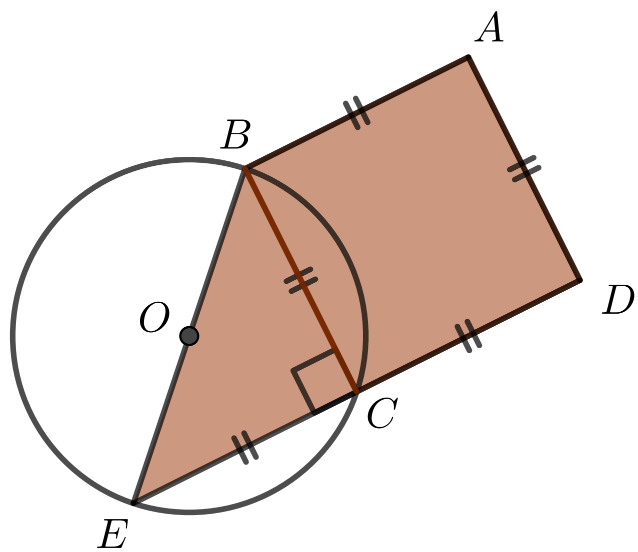
Problem of the Week

Problem C

A Circle and Other Shapes

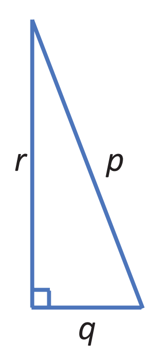
Quadrilateral $ABED$ is made up of square $ABCD$ and right isosceles $\triangle BCE$. BE is a diameter of the circle with centre O. Point C is also on the circle.

If the area of $ABED$ is 24 cm^2, what is the length of BE?

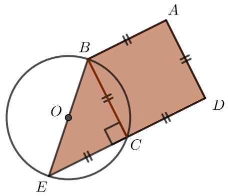


The *Pythagorean Theorem* states, "In a right triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides".

In the following right triangle, $p^2 = r^2 + q^2$.



Theme Geometry

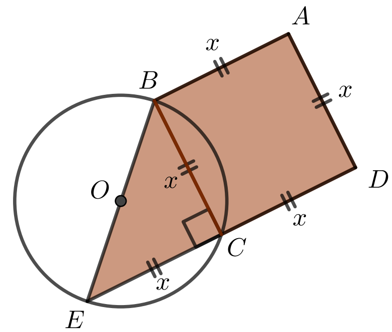


Problem of the Week
Problem C and Solution
A Circle and Other Shapes

Problem

Quadrilateral $ABED$ is made up of square $ABCD$ and right isosceles $\triangle BCE$. BE is a diameter of the circle with centre O. Point C is also on the circle. If the area of $ABED$ is 24 cm^2, what is the length of BE?

Solution



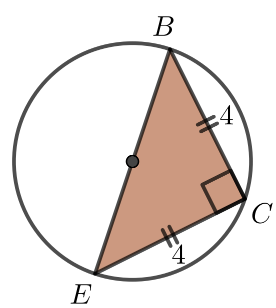
Let $AB = AD = DC = CB = CE = x$.
Therefore, the area of square $ABCD$ is x^2 and the area of $\triangle BCE$ is $\frac{1}{2}(x)(x) = 0.5x^2$.

Therefore,

$$\begin{aligned} \text{total area of quadrilateral } ABED &= \text{area of square } ABCD + \text{ area of } \triangle BCE \\ &= x^2 + 0.5x^2 \\ &= 1.5x^2 \end{aligned}$$

Now we also know that the area of $ABED$ is 24 cm^2. Therefore,

$$\begin{aligned} 1.5x^2 &= 24 \\ \frac{1.5x^2}{1.5} &= \frac{24}{1.5} \\ x^2 &= 16 \\ x &= 4, \text{ since } x > 0 \end{aligned}$$



Now let's look at $\triangle BCE$.
We know $BC = CE = 4$.
Using the Pythagorean Theorem,

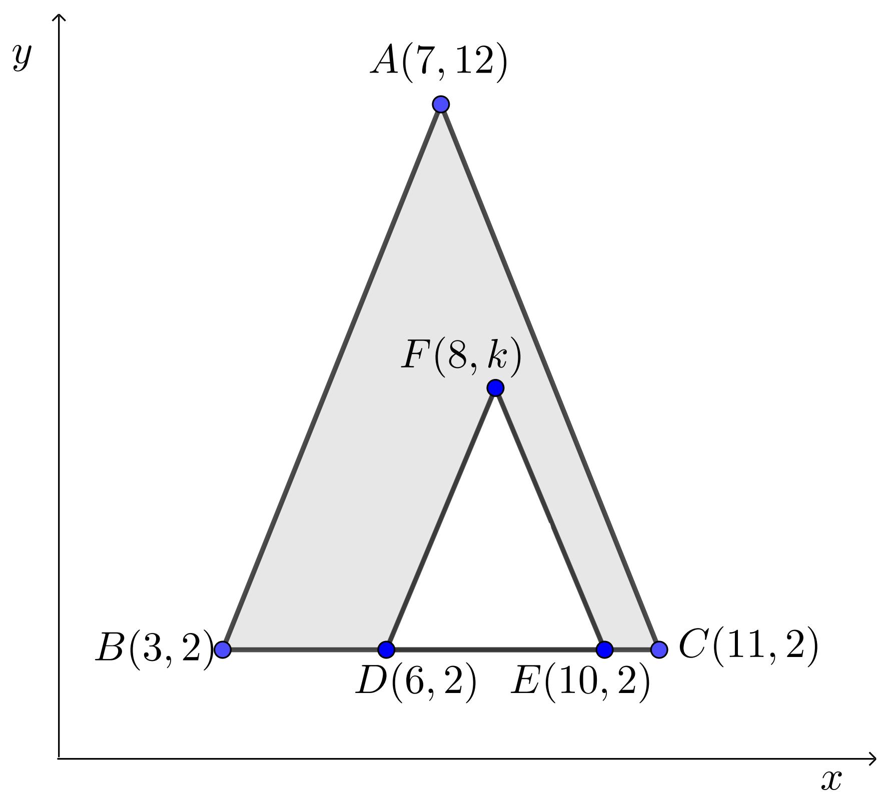
$$\begin{aligned} BE^2 &= 4^2 + 4^2 \\ &= 16 + 16 \\ &= 32 \\ BE &= \sqrt{32}, \text{ since } BE > 0 \end{aligned}$$

Therefore, $BE = \sqrt{32}$ cm, or approximately 5.7 cm.

Problem of the Week
Problem C
Up to a Certain Point

Points $A(7, 12)$, $B(3, 2)$, $C(11, 2)$, $D(6, 2)$ and $E(10, 2)$ are placed on the Cartesian plane, as shown below. The point F is placed inside $\triangle ABC$ so that the area of the shaded region is 32 units^2.

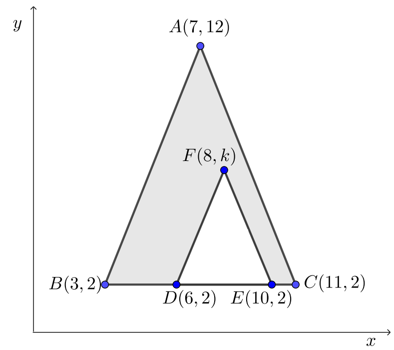
If the x-coordinate of F is 8, what is the y-coordinate of F?



Problem of the Week
Problem C and Solution
Up to a Certain Point

Problem

Points $A(7,12)$, $B(3,2)$, $C(11,2)$, $D(6,2)$ and $E(10,2)$ are placed on the Cartesian plane, as shown to the right. The point F is placed inside $\triangle ABC$ so that the area of the shaded region is 32 units^2. If the x-coordinate of F is 8, what is the y-coordinate of F?

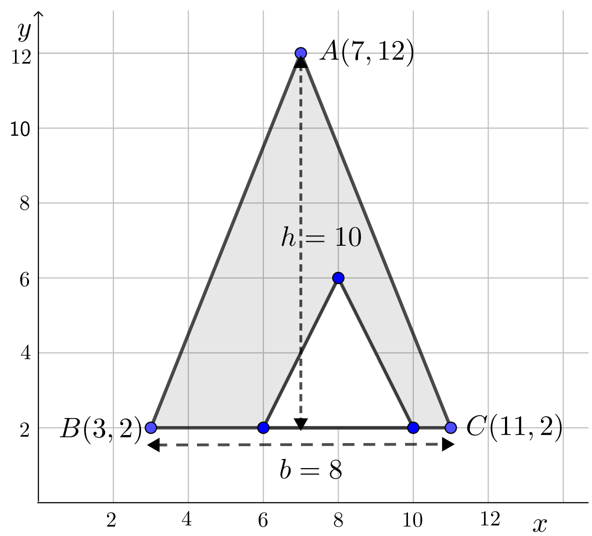


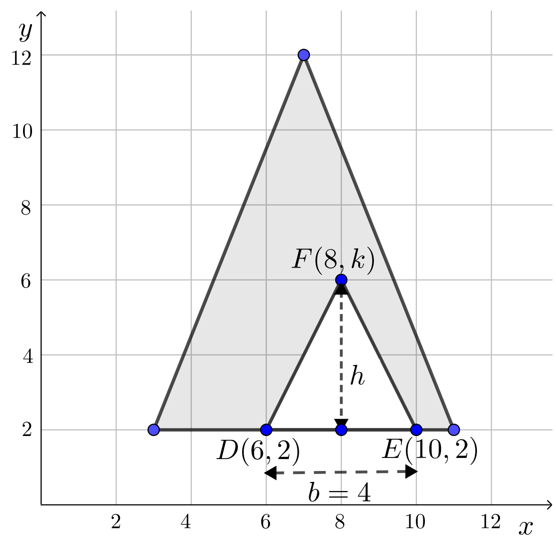
Solution

We will solve this problem in two ways. The first solution uses grid lines, the second solution does not. In both solutions, to find the area of the shaded region we will take the area of the large triangle, $\triangle ABC$, subtract the area of the small triangle, $\triangle DEF$, and then use the given information that this area is equal to 32 units^2.

Solution 1

To determine the height (h) and base (b) of each triangle, we use grid paper:





The area of $\triangle ABC = \frac{b\times h}{2} = \frac{8\times 10}{2} = 40$ (see above left).
The area of $\triangle DEF = \frac{b\times h}{2} = \frac{4\times h}{2} = 2h$ (see above right).

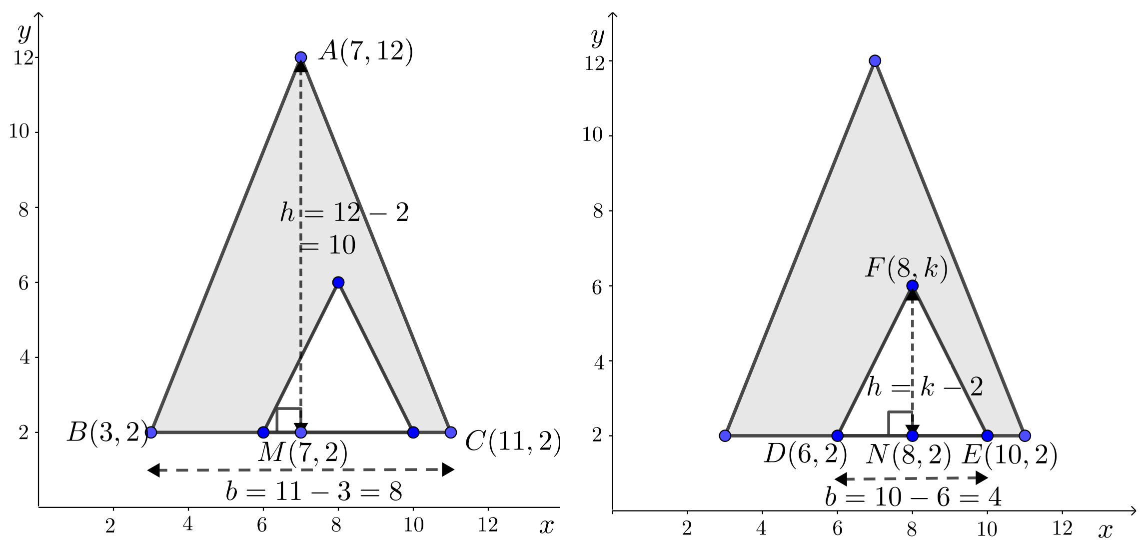
Therefore, the area of the shaded region is $40 - 2h$, which is also equal to 32. So $2h$ must equal 8, and therefore $h = 4$.

Now point F is $h = 4$ units higher than the base of the triangle, which is 2 units above the x-axis.
Therefore, the y-coordinate of point F is $2 + 4 = 6$.

Solution 2

In this solution, we use the fact that the distance between two points that have the same x-coordinate is the positive difference between their y-coordinates. We will also use the fact that the distance between two points that have the same y-coordinate is the positive difference between their x-coordinates.

In $\triangle ABC$, drop a perpendicular from vertex A to M on BC. Since BC is horizontal, then AM is vertical. Since every point on a vertical line has the same x-value, M has x-coordinate 7. Similarly, since M is on the horizontal line through $B(3, 2)$ and $C(11, 2)$, M has y-coordinate 2. Therefore, the base of $\triangle ABC$ is $b = 11 - 3 = 8$ and the height is $h = 12 - 2 = 10$. Now, the area of $\triangle ABC = \frac{b \times h}{2} = \frac{8 \times 10}{2} = 40$ (see below left).



In $\triangle DEF$, drop a perpendicular from vertex F to N on DE. Since DE is horizontal, then FN is vertical. Since every point on a vertical line has the same x-value, N has x-coordinate 8. Similarly, since N is on the horizontal line through $D(6, 2)$ and $E(10, 2)$, N has y-coordinate 2. Therefore, the base of $\triangle DEF$ is $b = 10 - 6 = 4$ and the height is $h = k - 2$. Now, the area of $\triangle DEF = \frac{b \times h}{2} = \frac{4 \times (k-2)}{2} = 2(k - 2)$ (see above right).

We can now solve for k:

$$\begin{aligned}
40 - 2(k-2) &= 32 \\
\text{Subtracting 32 from each side:} \quad 8 - 2(k-2) &= 0 \\
\text{Adding } 2(k-2) \text{ to each side:} \quad 8 &= 2(k-2) \\
4 &= k - 2 \\
6 &= k
\end{aligned}$$

Therefore, the y-coordinate of point F is 6.

Algebra (A)

TAKE ME TO THE COVER

Problem of the Week
Problem C
Burger Boxes

At the local burger restaurant, Patty always cooks burgers one at a time. After cooking a burger, she places it into one of three different boxes: one with dots, one with stripes, and one plain box. The first burger she places in a box with dots and puts it on the top of the stack. The second burger she places in a box with stripes and puts it on the top of the stack. The third burger she places in a plain box and puts it on the top of the stack. If she has cooked three burgers, she would have a stack as shown in Image 1.

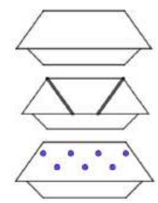


Image 1

Patty then repeats the pattern of placing the burgers in boxes. As she cooks a burger, she places that box on the top of the stack of not yet sold burgers, and continues to cycle through the three different boxes (dot, stripe, plain, dot, stripe, plain, ...) into which to place the burger. If she cooked two more burgers, she would have a stack as shown in Image 2.

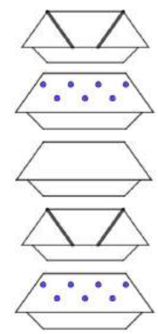


Image 2

At the same time, Tom sell burgers one at a time and always takes the uppermost box from the stack. Patty cooks faster than Tom can sell the burgers.

Shortly into a new shift, Patty has cooked some burgers and Tom has sold some burgers. The stack of unsold burgers looks like the stack in Image 3. What is the fewest number of burgers sold by Tom? Explain your answer.

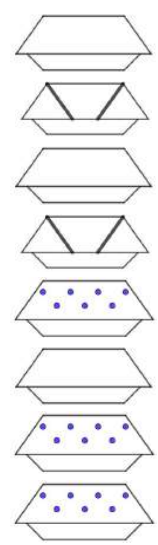
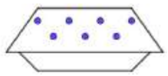
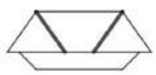


Image 3

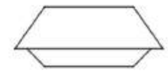
Themes Algebra, Computational Thinking

Problem of the Week

Problem C and Solution

Burger Boxes



Problem

At the local burger restaurant, Patty always cooks burgers one at a time. After cooking a burger, she places it into one of three different boxes: one with dots, one with stripes, and one plain box. If she has cooked three burgers, she would have a stack as shown in Image 1. Patty then repeats the pattern of placing the burgers in boxes. As she cooks a burger, she places that box on the top of the stack of not yet sold burgers, and continues to cycle through the three different boxes (dot, stripe, plain, dot, stripe, plain, ...) into which to place the burger. If she cooked two more burgers, she would have a stack as shown in Image 2. At the same time, Tom sell burgers one at a time and always takes the uppermost box from the stack. Patty cooks faster than Tom can sell the burgers. Shortly into a new shift, Patty has cooked some burgers and Tom has sold some burgers. The stack of unsold burgers looks like the stack in Image 3. What is the fewest number of burgers sold by Tom? Explain your answer.

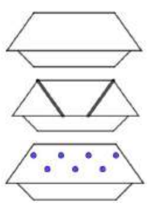


Image 1

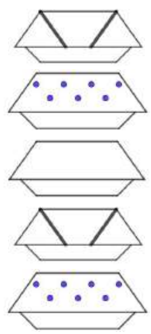


Image 2

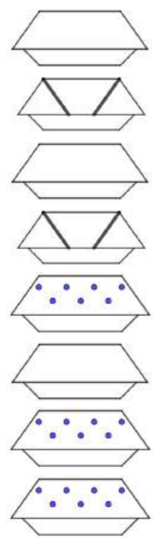


Image 3

Solution

Consider labelling the boxes as D (for dotted), S (for striped) and P (for plain, meaning no dots or stripes). So, we want to get a sequence that looks like: D, D, P, D, S, P, S, P where the leftmost boxes are at the bottom of the stack, and the rightmost boxes are at the top.

If no burgers were sold at all, we would have the sequence: $D, S, P, D, S, P, D, S, P, D, S, P, \ldots$

Notice that if we sell the burgers that are bolded as shown in the sequence:
$D, \mathbf{S}, \mathbf{P}, D, \mathbf{S}, P, D, S, P, \mathbf{D}, S, P,$ we have the sequence: D, D, P, D, S, P, S, P. This is the sequence that we are looking for. We can think of each bolded term as a burger being sold. The minimum number of boxes between the first two Ds is 2 (S, P). The minimum number of boxes between D and P is 1 (S). The minimum number boxes between the P and S is 1 (D). Therefore, the least amount of burgers sold is $2 + 1 + 1 = 4$.

The Beaver Computing Challenge (BCC):

This problem is based on a previous BCC problem. The BCC is designed to get students with little or no previous experience excited about computing. Questions are inspired by topics in computer science and connections to Computer Science are described in the solutions to all past BCC problems. If you enjoyed this problem, you may want to explore the BCC contest further.

Connections to Computer Science:

Computer scientists are concerned about how to efficiently store information. For certain problems, the best way to store information is in a data structure called a stack. A stack is a data structure that imposes the following rule about accessing data:

- new items can be put on the top of the stack (to become the new top of the stack): this is called pushing the element onto the stack
- items that are to be removed are removed from the top of the stack (making the element just below the top the new top of the stack): this is called popping the stack

Stacks are used for a variety of problem solving techniques, and perhaps the easiest one to visualize is the balanced parentheses problem. You would like to verify that some mathematical expression involving parentheses is valid. So (1+1), is valid, ((2+3)*(1+1)) is valid and so on. Ignoring any of the numbers or operators, we can ensure that we have a valid sequence of parentheses by the following simple process:

- read the mathematical expression from left-to-right;
- if we see (, push (onto the stack;
- if we see), pop the top (symbol from the stack;
- if we try to pop an empty stack, i.e., a stack without anything on it, the sequence is invalid;
- if we read the entire mathematical expression and the stack is not empty, the sequence is invalid;
- otherwise, the sequence is valid.

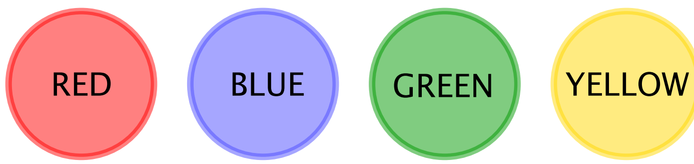
You can verify that the sequences above are verified by this algorithm, and that sequences like (((1+1) and)()()(would be correctly determined to be invalid by this algorithm.

Problem of the Week
Problem C
Playing with Disks

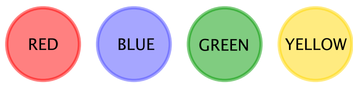
A bag contains circular disks. In the bag, there are 5 blue disks, 6 red disks, 3 green disks and 2 yellow disks. Several orange disks are added to the bag. All the disks in the bag are identical except for colour.

A disk is now randomly selected from the bag. The probability of a blue or green disk being selected is now $\frac{2}{7}$.

How many orange disks were added to the bag?



Themes Algebra, Data Management



Problem of the Week
Problem C and Solution
Playing with Disks

Problem

A bag contains circular disks. In the bag, there are 5 blue disks, 6 red disks, 3 green disks and 2 yellow disks. Several orange disks are added to the bag. All the disks in the bag are identical except for colour. A disk is now randomly selected from the bag. The probability of a blue or green disk being selected is now $\frac{2}{7}$. How many orange disks were added to the bag?

Solution

Solution 1

At present there are $5 + 6 + 3 + 2 = 16$ disks in the bag. Since, after adding some orange disks to the bag, the probability of picking a blue or green disk is $\frac{2}{7}$, this implies that the new total number of disks in the bag is a multiple of 7 and this multiple must be greater than 16. Therefore, there are possibly 21, 28, 35, 42, 49, 56, $\cdots$ disks in the bag.

The number of blue and green disks in the bag is $5 + 3 = 8$. If k is the total number of disks in the bag after adding some orange disks, then $\frac{8}{k} = \frac{2}{7}$ but $\frac{2}{7} = \frac{8}{28}$ so $\frac{8}{k} = \frac{8}{28}$ and $k = 28$ follows.

Since there were 16 disks in the bag and there are now 28 disks in the bag, then $28 - 16 = 12$ orange disks were added to the bag.

Therefore, 12 orange disks were added to the bag.

Solution 2

Let w be the number of orange disks added to the bag.

There are now $5 + 6 + 3 + 2 + w = 16 + w$ disks in the bag. The number of blue and green disks in the bag is $5 + 3 = 8$. We know that the number of blue and green disks divided by the total number of disks in the bag is $\frac{2}{7}$, so $\frac{8}{16+w} = \frac{2}{7}$.

If we multiply the numerator and denominator of the fraction $\frac{2}{7}$ by 4, we obtain $\frac{2}{7} = \frac{2\times4}{7\times4} = \frac{8}{28}$, which has the same numerator as the fraction $\frac{8}{16+w}$.

Since the fractions are equal and the numerators are equal, the denominators must also be equal. It follows that $16 + w = 28$ and $w = 12$.

Therefore, 12 orange disks were added to the bag.

Problem of the Week
Problem C
Vote!

In an election for a grade representative on the school council, there were only two candidates.

Freda First received 60% of the total votes and Saheel Second received all the rest. If Freda won by 28 votes, how many people voted?

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Vote!

Problem

In an election for a grade representative on the school council, there were only two candidates. Freda First received 60% of the total votes and Saheel Second received all the rest. If Freda won by 28 votes, how many people voted?

Solution

If Freda received 60% of the votes, then Saheel received 40% of the total number of votes. The difference between them, 20%, represents 28 votes.

We are interested in determining 100% of the votes, that is, the total number of votes cast. Since we know that 20% of the total votes cast was 28 votes and $5 \times 20 = 100$, then the total number of votes cast was 5×28 or 140 votes.

Solution 2

The second solution uses an algebraic approach.

Let n represent the total number of votes cast.
Since Freda received 60% of the total votes, she received $0.6 \times n$ or $0.6n$ votes.
Since Saheel received all of the remaining votes, he received $0.4 \times n$ or $0.4n$ votes.

We know that the difference between the number of votes received by Freda and the number of votes received by Saheel was 28. So,

$$\begin{aligned} 0.6n - 0.4n &= 28 \\ 0.2n &= 28 \\ \frac{0.2n}{0.2} &= \frac{28}{0.2} \\ n &= 140 \end{aligned}$$

Therefore, there were 140 votes cast.

Problem of the Week

Problem C

Thinking About Products

Three cards are lined up on a table. Each card has a letter printed on one side and a positive number printed on the other side. One card has an R printed on it, one card has a G printed on it, and one card has a B printed on it. The number side of each card is facedown on the table. The following is known about the three concealed numbers:

(i) the product of the number on the card with an R and the number on the card with a G equals the number on the card with a B;

(ii) the product of the number on the card with a G and the number on the card with a B is 180; and

(iii) five times the number on the card with a B equals the number on the card with a G.

Determine the product of the numbers on the three cards.

THEMES ALGEBRA, NUMBER SENSE

Problem of the Week
Problem C and Solution
Thinking About Products

Problem

Three cards are lined up on a table. Each card has a letter printed on one side and a positive number printed on the other side. One card has an R printed on it, one card has a G printed on it, and one card has a B printed on it. The number side of each card is facedown on the table. The product of the number on the card with an R and the number on the card with a G equals the number on the card with a B. The product of the number on the card with a G and the number on the card with a B is 180. And, five times the number on the card with a B equals the number on the card with a G. Determine the product of the numbers on the three cards.

Solution

Solution 1

Let the three numbers be represented by r, g, and b.

Since the product of the number on the card with an R and the number on the card with a G equals the number on the card with a B, $r \times g = b$. We are looking for $r \times g \times b = (r \times g) \times b = (b) \times b = b^2$. So when we find b^2 we have found the required product $r \times g \times b$.

We are also given that that $g \times b = 180$ and $g = 5 \times b$, so $g \times b = 180$ becomes $(5 \times b) \times b = 180$ or $5 \times b^2 = 180$. Dividing by 5, we obtain $b^2 = 36$. This is exactly what we are looking for since $r \times g \times b = b^2$. Therefore, the product of the three numbers is 36.

For those who need to know what the actual numbers are, we can proceed and find the three numbers. We know $b^2 = 36$, so $b = 6$ since b is a positive number. Therefore, $g = 5 \times b = 5 \times 6 = 30$. And finally, $r \times g = b$ so $r \times (30) = 6$. Dividing by 30, we get $r = \frac{6}{30} = \frac{1}{5}$. We can verify the product $r \times g \times b = (\frac{1}{5}) \times (30) \times (6) = 6 \times 6 = 36$.

Solution 2

In this solution we will try to find the numbers by working with the factors of 180.

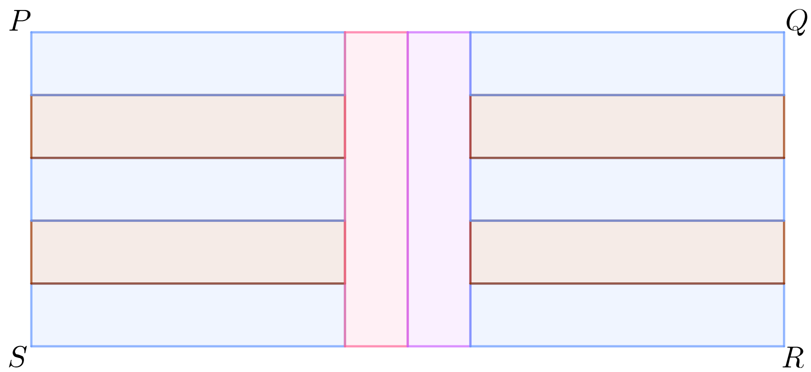
The product of the number on the card with a G and the number on the card with a B is 180 and the number on the card with a G is five times the number on the card with a B. The number 180 can be written as $2 \times 2 \times 3 \times 3 \times 5$. By playing with the factors, we can get the number on the card with a G is $5 \times 2 \times 3$ and the number on the card with a B is 2×3. That is, the number on the card with a G could be 30 and the number on the card with a B could be 6.

Now using the fact that the number on the card with an R times the number on the card with a G is equal to the number on the card with a B, we see that some number times 30 equals 6 and it follows that the number on the card with an R would be $6 \div 30 = \frac{1}{5}$.

The product of the three numbers is $\frac{1}{5} \times 30 \times 6 = 6 \times 6 = 36$.

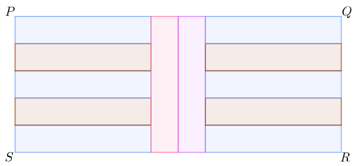
This solution only works because the number on the card with a G and the number on the card with a B happen to be integers.

Problem of the Week
Problem C
A Perfect Dozen



Twelve identical rectangles are arranged as shown in the diagram to form a large rectangle $PQRS$.

If the area of rectangle $PQRS$ is 540 cm^2, determine the dimensions of the smaller rectangles.



Problem of the Week
Problem C and Solution
A Perfect Dozen

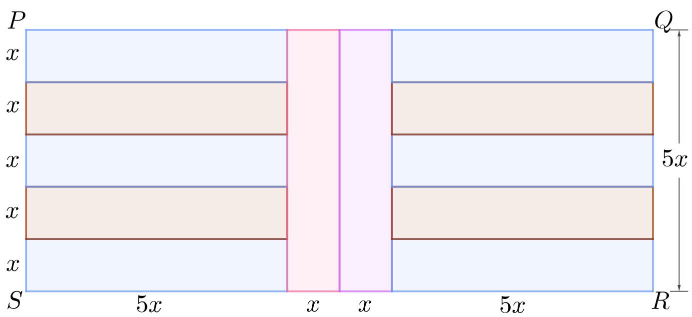
Problem

Twelve identical rectangles are arranged as shown in the diagram above to form a large rectangle $PQRS$. If the area of rectangle $PQRS$ is 540 cm^2, determine the dimensions of the smaller rectangles.

Solution

Solution 1

Let x be the width of one of the smaller identical rectangles, in cm. Five of the smaller rectangles are stacked on top of each other creating PS, so $PS = x + x + x + x + x = 5x$. Since $PQRS$ is a rectangle, $PS = QR = 5x$. But $5x$ then also is the length of a smaller rectangle. Therefore, a smaller rectangle is $5x$ cm by x cm. This information is all marked on the following diagram.



The area of rectangle $PQRS$ is the same as 12 times the area of one of the smaller rectangles.

$$\begin{aligned} \text{Area } PQRS &= 12 \times \text{Area of one smaller rectangle} \\ 540 &= 12 \times 5x \times x \\ 540 &= 60 \times x^2 \end{aligned}$$

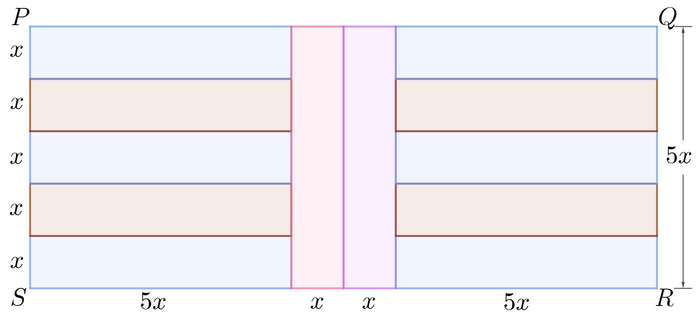
Dividing both sides by 60, we obtain $x^2 = 9$ and $x = 3$ follows. ($x > 0$ since x is the width of a smaller rectangle.)

The width of a smaller rectangle is $x = 3$ cm and the length of a smaller rectangle is $5x = 5(3) = 15$ cm.

Therefore, the smaller rectangles are each 15 cm long and 3 cm wide.

Solution 2

Let x be the width of one of the smaller identical rectangles, in cm. Five rectangles are stacked on top of each other creating PS, so
$PS = x + x + x + x + x = 5x$.



Also $PQRS$ is a rectangle so $QR = PS = 5x$ and $5x$ is also the length of a smaller rectangle.

Now SR is made up of the lengths of two of the smaller rectangles plus the widths of two of the smaller rectangles. Therefore, $SR = 5x + 5x + x + x = 12x$ and rectangle $PQRS$ is $12x$ cm long and $5x$ cm wide.

To find the area of $PQRS$ we multiply the length SR by the width QR.

$$\begin{aligned} \text{Area } PQRS &= SR \times QR \\ 540 &= (12x) \times (5x) \\ 540 &= 12 \times 5 \times x \times x \\ 540 &= 60 \times x^2 \end{aligned}$$

Dividing by 60, we obtain $x^2 = 9$ and $x = 3$ follows. ($x > 0$ since x is the width of a smaller rectangle.)

The width of a smaller rectangle is $x = 3$ cm and the length of a smaller rectangle is $5x = 5(3) = 15$ cm.

Therefore, the smaller rectangles are each 15 cm long and 3 cm wide.

Problem of the Week

Problem C

Location, Location, Location

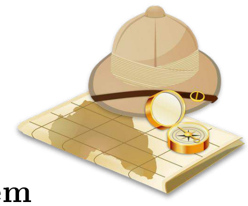
Starting with 2, we will place the integers as shown in the following chart.

	U	V	W	X	Y
Row 1			2	3	4
Row 2	7	6	5		
Row 3			8	9	10
Row 4	13	12	11		
Row 5			14	15	16
Row 6	19	18	17		

The pattern continues. Each new row lists the next three integers, in the direction opposite to the row above, and with the smallest of the three integers in column W.

Determine the exact location of the integer 2020. State the row number and the column letter (U, V, W, X, Y).

Themes Algebra, Number Sense



Problem of the Week

Problem C and Solution

Location, Location, Location

Problem

Starting with 2, we will place the integers as shown in the following chart.

	U	V	W	X	Y
Row 1			2	3	4
Row 2	7	6	5		
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Row 4	13	12	11		
Row 5			14	15	16
Row 6	19	18	17		

The pattern continues. Each new row lists the next three integers, in the direction opposite to the row above, and with the smallest of the three integers in column W.

Determine the exact location of the integer 2020. State the row number and the column letter (U, V, W, X, Y).

Solution

Observe some of the patterns in the chart. (There are many more patterns than the ones listed below.)

Each row contains a multiple of three in either column V or column X. Even multiples of three are in column V and odd multiples of three are in column X. To determine the row number, take the multiple of three and divide it by 3.

The outer numbers in column U or column Y have a remainder 1 when divided by 3. Numbers that are even and have a remainder 1 when divided by 3 are in column Y. Numbers that are odd and have a remainder 1 when divided by 3 are in column U. If the largest number in a row is even, it is in column Y. If the largest number in a row is odd, it is in column U.

Every number in column W has a remainder 2 when divided by 3.

When 2020 is divided by 3, there is a quotient of 673 and a remainder 1. So 2020 is in column U or Y but since 2020 is even, it is in column Y. The 673^{rd} multiple of 3, which is 2019, is in row 673, in column X, to the left of 2020. In fact, row 673 will contain 2018 in column W; 2019, the 673^{rd} multiple of 3, in column X; and 2020 in column Y. Therefore, the number 2020 is located in row 673, column Y.

Problem of the Week
Problem C
Accumulating Change

Canadian one-dollar coins are referred to as *loonies*. Canadian two-dollar coins are referred to as *toonies*.

Penny Saver has been saving quarters, loonies and toonies in her coin bank for a long time. No other types of coins are in her bank. One third of the coins in the bank are quarters and one fifth of the coins are loonies. There are 56 toonies in the bank.

Determine how much money Penny has saved in her coin bank.

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Accumulating Change

Problem
Penny Saver has been saving quarters, loonies and toonies in her coin bank for a long time. No other types of coins are in her bank. One third of the coins in the bank are quarters and one fifth of the coins are loonies. There are 56 toonies in the bank. Determine how much money Penny has saved in her coin bank.

Solution
Solution 1

One of the key sentences in the problem is "No other types of coins are in her bank." Using the fractions given, it will be possible to determine what fraction of the whole is made up by toonies.

$$\text{The fraction of toonies in the bank} = 1 - \frac{1}{3} - \frac{1}{5} = \frac{15}{15} - \frac{5}{15} - \frac{3}{15} = \frac{7}{15}.$$

We can now determine the total number of coins in the bank. Since 56 toonies are in the bank and $\frac{7}{15}$ of the coins are toonies,

$$\begin{array}{rrcl} & \frac{7}{15} \text{ of the coins in the bank} & = & 56 \text{ coins.} \\ \text{Dividing by 7,} & \frac{1}{15} \text{ of the coins in the bank} & = & 56 \div 7 = 8 \text{ coins.} \\ \text{Multiplying by 15,} & \frac{15}{15} \text{ of the coins in the bank} & = & 8 \times 15 = 120 \text{ coins.} \end{array}$$

There are 120 coins in the bank. We can now determine the number of quarters and loonies. The number of quarters $= \frac{1}{3} \times 120 = 40$ and the number of loonies $= \frac{1}{5} \times 120 = 24$.

To determine the amount of money in the bank, we multiply the value of a particular coin by the quantity of that coin and add the three values together.

$$\begin{array}{rcl} \text{Amount in the Bank} & = & \text{Value of Quarters} + \text{Value of Loonies} + \text{Value of Toonies} \\ & = & \$0.25 \times 40 \quad + \quad \$1.00 \times 24 \quad + \quad \$2 \times 56 \\ & = & \$10.00 \quad + \quad \$24.00 \quad + \quad \$112.00 \\ & = & \$146.00 \end{array}$$

Therefore, Penny has a total of \$146 in her bank.

A second solution, using equations, is found on the next page.

Solution 2

Let C represent the number of coins in the bank.

Then $\frac{1}{3}C$ is the number of quarters in the bank and $\frac{1}{5}C$ is the number of loonies in the bank.

It follows that $C - \frac{1}{3}C - \frac{1}{5}C = \frac{15}{15}C - \frac{5}{15}C - \frac{3}{15}C = \frac{7}{15}C$ is the number of toonies in the bank.

But there are 56 toonies in the bank, so

$$\begin{aligned} \frac{7}{15}C &= 56 & \\ \frac{1}{15}C &= 8 & \text{(after dividing both sides by 7)} \\ C &= 120 & \text{(after multiplying both sides by 15)} \end{aligned}$$

There are 120 coins in the bank.

Then $\frac{1}{3} \times 120 = 40$ coins are quarters and $\frac{1}{5} \times 120 = 24$ coins are loonies.

To determine the amount of money in the bank, we multiply the value of a particular coin by the quantity of that coin and add the three values together.

$$\begin{aligned} \text{Amount in the Bank} &= \text{Value of Quarters} + \text{Value of Loonies} + \text{Value of Toonies} \\ &= \$0.25 \times 40 + \$1.00 \times 24 + \$2 \times 56 \\ &= \$10.00 + \$24.00 + \$112.00 \\ &= \$146.00 \end{aligned}$$

Therefore, Penny has a total of \$146 in her bank.

Problem of the Week

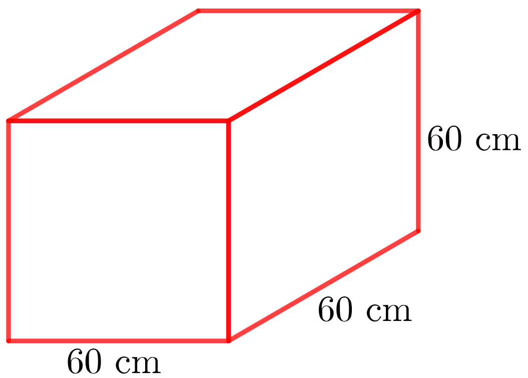
Problem C

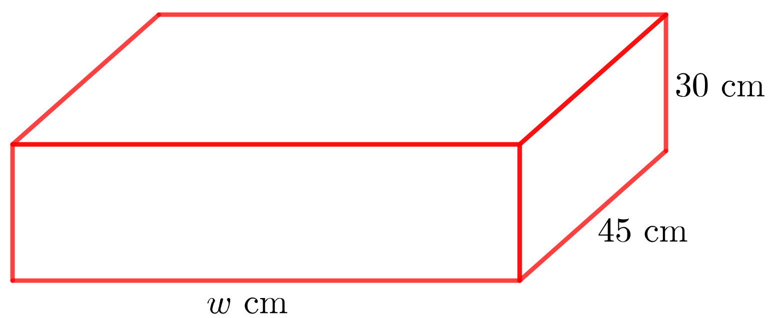
Can You Build It?

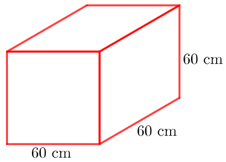
Two students are building storage boxes in woodworking class. The first student builds a box in the form of a cube with each edge measuring 60 cm. The second student builds a box in the shape of a rectangular prism with width w cm, height 30 cm and depth 45 cm.

Surprisingly, each student's box has the same total surface area.

Determine which student's box has the greatest volume. How much greater is the volume?



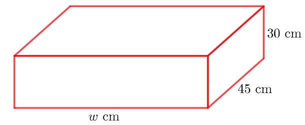




Problem of the Week

Problem C and Solution

Can You Build It?



Problem

Two students are building storage boxes in woodworking class. The first student builds a box in the form of a cube with each edge measuring 60 cm. The second student builds a box in the shape of a rectangular prism with width w cm, height 30 cm and depth 45 cm. Surprisingly, each student's box has the same total surface area. Determine which student's box has the greatest volume. How much greater is the volume?

Solution

To find the total surface area of a cube, determine the area of one side and multiply by 6. Therefore, the total surface area of the cube-shaped box is $6 \times 60 \times 60 = 21\,600$ cm^2.

To find the total surface area of the rectangular prism, determine the sum of the areas of the six sides. The front side and the back side have equal areas. The top and the bottom have equal areas. Each of the remaining two sides have equal area. Therefore, the total surface area of the rectangular prism box is

$$\begin{aligned} & 2 \times \text{area of front} + 2 \times \text{area of top} + 2 \times \text{area of side} \\ = \;& 2 \times 30 \times w + 2 \times 45 \times w + 2 \times 30 \times 45 \\ = \;& 60w + 90w + 2700 \\ = \;& 150w + 2700 \end{aligned}$$

But the total surface area of the cube-shaped box is the same as the total surface area of the rectangular prism. So,

$$\begin{aligned} 21\,600 &= 150w + 2700 \\ 21\,600 - 2700 &= 150w + 2700 - 2700 \\ 18\,900 &= 150w \\ \frac{18\,900}{150} &= \frac{150w}{150} \\ 126 &= w \end{aligned}$$

Therefore, the width of the rectangular prism is 126 cm.

To find the volume of a cube, "cube" the edge length. So, the volume of the cube is $60 \times 60 \times 60 = 60^3 = 216\,000$ cm^3.

To find the volume of the rectangular prism, multiply the three different edge lengths. So, the volume of the rectangular prism is $126 \times 45 \times 30 = 170\,100$ cm^3.

Therefore, the student who is building the cube has the box with the greater volume. The volume of the cube is greater by $216\,000 - 170\,100 = 45\,900$ cm^3.

Problem of the Week
Problem C
Sum of Everything

If you were to list the integers from 1 to 12, you would get the list

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$$

If you were to sum the digits of the integers in this list, you would get the sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + (1 + 0) + (1 + 1) + (1 + 2) = 51.$$

Below are the integers from 1 to 100. Can you find the sum of all of the digits of these numbers?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

THEMES ALGEBRA, NUMBER SENSE

Problem of the Week
Problem C and Solution
Sum of Everything

Problem

If you were to list the integers from 1 to 12, you would get the list $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.
If you were to sum the digits of the integers in this list, you would get the sum
$1+2+3+4+5+6+7+8+9+(1+0)+(1+1)+(1+2) = 51$.

To the right are the integers from 1 to 100. Can you find the sum of all of the digits of these numbers?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Solution

(1) In the table above, each of the ten columns has a units digit that occurs ten times. So the sum of ALL of the units digits is

$$\begin{aligned}&10(1)+10(2)+10(3)+10(4)+10(5)+10(6)+10(7)+10(8)+10(9)+10(0)\\&= 10(1+2+3+4+5+6+7+8+9+0)\\&= 10(45)\\&= 450\end{aligned}$$

(2) Each of the ten columns has a tens digit from 0 to 9. So the sum of ALL of the tens digits is

$$\begin{aligned}&10(0+1+2+3+4+5+6+7+8+9)\\&= 10(45)\\&= 450\end{aligned}$$

(3) The number 100 is the only number with a hundreds digit. We need to add 1 to our final sum.

(4) Now we add our results from (1), (2), and (3) to obtain the required sum.

$$\begin{aligned}\text{Sum of digits} &= \text{Units digit sum} + \text{Tens digit sum} + \text{Hundreds Digit}\\&= 450+450+1\\&= 901\end{aligned}$$

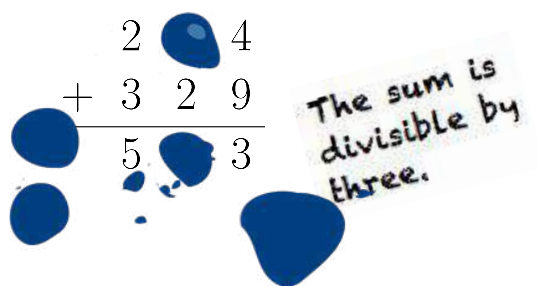
Therefore, the sum of all of the digits of the numbers from 1 to 100 is 901.

Problem of the Week

Problem C

What a Mess!

Jing was looking through an old math notebook and found the following mess on one of the pages:



The middle digit of the top and bottom numbers were both smudged and unreadable. Beside the question there was a handwritten note which read: "The sum is divisible by three."

If the missing middle digit in the top number is A and the missing middle digit in the bottom number is B, determine all possible values for A and B.

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
What a Mess!

Problem

Jing was looking through an old math notebook and found the following question on one of the pages. The middle digit of the top and bottom numbers were both smudged and unreadable. Beside the question there was a handwritten note which read: "The sum is divisible by three." If the missing middle digit in the top number is A and the missing middle digit in the bottom number is B, determine all possible values for A and B.

$$\begin{array}{cccc} & 2 & A & 4 \\ + & 3 & 2 & 9 \\ \hline & 5 & B & 3 \end{array}$$

Solution

Solution 1

First we find the possible values of B. For a number to be divisible by 3, the sum of its digits must be divisible by 3. So $5 + B + 3$ must be divisible by 3. The only possible values for B are thus 1, 4, or 7.

We know that $2A4 + 329 = 5B3$ so $2A4 = 5B3 - 329$.

We can try each of the possible values for B in the equation $2A4 = 5B3 - 329$ to find values of A that make the equation true.

1. If $B = 1$, then $513 - 329 = 184$, which cannot equal $2A4$. So when $B = 1$ there is no A to satisfy the problem.
2. If $B = 4$, then $543 - 329 = 214$, which does equal $2A4$ when $A = 1$. So for $A = 1$ and $B = 4$ there is a valid solution.
3. If $B = 7$, then $573 - 329 = 244$, which does equal $2A4$ when $A = 4$. So for $A = 4$ and $B = 7$ there is a valid solution.

Therefore, when $A = 1$ and $B = 4$ or when $A = 4$ and $B = 7$, the given problem has a valid solution.

Solution 2

$$\begin{array}{cccc} & 2 & A & 4 \\ + & 3 & 2 & 9 \\ \hline & 5 & B & 3 \end{array}$$

When the digits in the unit's column are added together, there is one carried to the ten's column. When the digits in the hundred's column are added together we get $2 + 3 = 5$ so there is no carry from the ten's column. Therefore, when the ten's column is added we get $1 + A + 2 = B$ or $A + 3 = B$.

We can now look at all possible values for A that produce a single digit value for B in the number $5B3$. We can then determine whether or not $5B3$ is divisible by 3.

The following table summarizes the results.

A	$B = A + 3$	$5B3$	Divisible by 3 (*yes*/*no*)?
0	3	533	*no*
1	4	543	*yes*
2	5	553	*no*
3	6	563	*no*
4	7	573	*yes*
5	8	583	*no*
6	9	593	*no*

Therefore, when $A = 1$ and $B = 4$ or when $A = 4$ and $B = 7$, the given problem has a valid solution.

Problem of the Week

Problem C

We Took the Cookies

There is a cookie jar that contains a certain number of cookies. Three friends divide the cookies in the following way. First, Harold takes all of the cookies and places them into three equal piles with none left over. He then keeps one of the piles and puts the other two piles back into the jar. Next, Lucie takes the remaining cookies in the jar and places them into three equal piles with none left over. She then keeps one of the piles and puts the other two piles back into the jar. Finally, Livio takes the remaining cookies in the jar and places them into three equal piles, but there is one cookie left over. He then keeps the leftover cookie and one of the piles and puts the other two piles back into the jar.

If there are now 10 cookies in the jar, how many cookies were originally in the cookie jar?

THEMES ALGEBRA, NUMBER SENSE

Problem of the Week
Problem C and Solution
We Took the Cookies

Problem

There is a cookie jar that contains a certain number of cookies. Three friends divide the cookies in the following way. First, Harold takes all of the cookies and places them into three equal piles with none left over. He then keeps one of the piles and puts the other two piles back into the jar. Next, Lucie takes the remaining cookies in the jar and places them into three equal piles with none left over. She then keeps one of the piles and puts the other two piles back into the jar. Finally, Livio takes the remaining cookies in the jar and places them into three equal piles, but there is one cookie left over. He then keeps the leftover cookie and one of the piles and puts the other two piles back into the jar. If there are now 10 cookies in the jar, how many cookies were originally in the cookie jar?

Solution

We will present two different solutions. Solution 1 works backwards through the problem. Solution 2 is an algebraic solution.

Solution 1

The last 10 cookies in the cookie jar are also the remaining two of Livio's three piles. Therefore, each pile he made had $10 \div 2 = 5$ cookies. Therefore, there were $5 \times 3 = 15$ cookies in the three piles he made plus 1 more cookie that he kept, for a total of 16 cookies. Therefore, there were 16 cookies in the cookie jar when Livio started dividing the cookies.

These 16 cookies were two of the three piles that Lucie made. Therefore, each pile that she made had $16 \div 2 = 8$ cookies. Therefore, there were $8 \times 3 = 24$ cookies in the three piles that she made. Therefore, there were 24 cookies in the cookie jar when Lucie started dividing the cookies.

These 24 cookies were two of the three piles that Harold made. Therefore, each pile that he made had $24 \div 2 = 12$ cookies. Therefore, there were $12 \times 3 = 36$ cookies in the three piles that he made. Therefore, there were 36 cookies in the cookie jar when Harold started dividing the cookies.

Therefore, there were originally 36 cookies in the cookie jar.

Solution 2

Let the initial number of cookies in the cookie jar be C.

Harold has $\frac{1}{3}C$ cookies in the pile he keeps. Therefore, $\frac{2}{3}C$ cookies are left for Lucie.

Lucie keeps $\frac{1}{3}$ of $\frac{2}{3}C$ cookies. Therefore, $\frac{2}{3}$ of $\frac{2}{3}C$ cookies are left for Livio. That is, $\frac{2}{3} \times \frac{2}{3}C = \frac{4}{9}C$ cookies are left for Livio.

For Livio, the pile he keeps is $\frac{1}{3}$ of one less than what is left. That is $\frac{1}{3} \times \left(\frac{4}{9}C - 1\right)$, and so the remaining number of cookies that he puts back into the cookie jar is equal to $\frac{2}{3} \times \left(\frac{4}{9}C - 1\right)$.

This is also equal to 10. That is,

$$\frac{2}{3} \times \left(\frac{4}{9}C - 1\right) = 10$$

Dividing both sides by $\frac{2}{3}$,

$$\frac{\frac{2}{3} \times \left(\frac{4}{9}C - 1\right)}{\frac{2}{3}} = \frac{10}{\frac{2}{3}}$$

Since $10 \div \frac{2}{3} = 10 \times \frac{3}{2} = \frac{30}{2} = 15$,

$$\frac{4}{9}C - 1 = 15$$

Therefore,

$$\frac{4}{9}C = 16$$

Dividing both sides by $\frac{4}{9}$,

$$\frac{\frac{4}{9}C}{\frac{4}{9}} = \frac{16}{\frac{4}{9}}$$

Since $16 \div \frac{4}{9} = 16 \times \frac{9}{4} = 36$,

$$C = 36$$

Therefore, there were originally 36 cookies in the cookie jar.

Data Management (D)

TAKE ME TO THE COVER

Problem of the Week

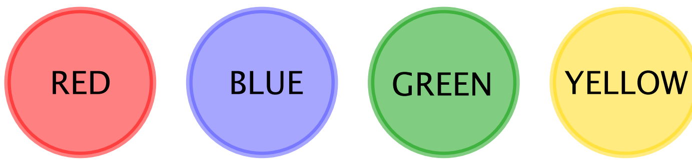
Problem C

Playing with Disks

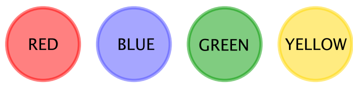
A bag contains circular disks. In the bag, there are 5 blue disks, 6 red disks, 3 green disks and 2 yellow disks. Several orange disks are added to the bag. All the disks in the bag are identical except for colour.

A disk is now randomly selected from the bag. The probability of a blue or green disk being selected is now $\frac{2}{7}$.

How many orange disks were added to the bag?



Themes Algebra, Data Management



Problem of the Week
Problem C and Solution
Playing with Disks

Problem

A bag contains circular disks. In the bag, there are 5 blue disks, 6 red disks, 3 green disks and 2 yellow disks. Several orange disks are added to the bag. All the disks in the bag are identical except for colour. A disk is now randomly selected from the bag. The probability of a blue or green disk being selected is now $\frac{2}{7}$. How many orange disks were added to the bag?

Solution

Solution 1

At present there are $5 + 6 + 3 + 2 = 16$ disks in the bag. Since, after adding some orange disks to the bag, the probability of picking a blue or green disk is $\frac{2}{7}$, this implies that the new total number of disks in the bag is a multiple of 7 and this multiple must be greater than 16. Therefore, there are possibly 21, 28, 35, 42, 49, 56, $\cdots$ disks in the bag.

The number of blue and green disks in the bag is $5 + 3 = 8$. If k is the total number of disks in the bag after adding some orange disks, then $\frac{8}{k} = \frac{2}{7}$ but $\frac{2}{7} = \frac{8}{28}$ so $\frac{8}{k} = \frac{8}{28}$ and $k = 28$ follows.

Since there were 16 disks in the bag and there are now 28 disks in the bag, then $28 - 16 = 12$ orange disks were added to the bag.

Therefore, 12 orange disks were added to the bag.

Solution 2

Let w be the number of orange disks added to the bag.

There are now $5 + 6 + 3 + 2 + w = 16 + w$ disks in the bag. The number of blue and green disks in the bag is $5 + 3 = 8$. We know that the number of blue and green disks divided by the total number of disks in the bag is $\frac{2}{7}$, so $\frac{8}{16+w} = \frac{2}{7}$.

If we multiply the numerator and denominator of the fraction $\frac{2}{7}$ by 4, we obtain $\frac{2}{7} = \frac{2\times 4}{7\times 4} = \frac{8}{28}$, which has the same numerator as the fraction $\frac{8}{16+w}$.

Since the fractions are equal and the numerators are equal, the denominators must also be equal. It follows that $16 + w = 28$ and $w = 12$.

Therefore, 12 orange disks were added to the bag.

Problem of the Week
Problem C
Missed Out

A class of 30 students went on a field trip to a nature centre. Throughout the day, two different workshops were offered: "Know Your Birds" and "Animals Around Us". Twelve of the students attended the workshop about birds. Seventeen of the students attended the "Animals Around Us" workshop. Five students attended both workshops. Some students did not attend any of the workshops.

How many students missed out by not attending any workshop during the field trip?

Problem of the Week

Problem C and Solution

Missed Out

Problem

A class of 30 students went on a field trip to a nature centre. Throughout the day, two different workshops were offered: “Know Your Birds” and “Animals Around Us”. Twelve of the students attended the workshop about birds. Seventeen of the students attended the “Animals Around Us” workshop. Five students attended both workshops. Some students did not attend any of the workshops. How many students missed out by not attending any workshop during the field trip?

Solution

Since 5 students attended both workshops and these students are included in the 12 who attended the “Know Your Birds” workshop, then $12 - 5$ or 7 students attended only the “Know Your Birds” workshop. They did not attend the “Animals Around Us”’ workshop.

Again, since 5 students attended both workshops and these students are included in the 17 who attended the “Animals Around Us” workshop, then $17 - 5$ or 12 students attended only the “Animals Around Us” workshop. They did not attend the “Know Your Birds” workshop as well.

Students will be in exactly one of four possible groups: they attended both workshops, they attended the “Know Your Birds” workshop only, they attended the “Animals Around Us” workshop only, or they did not attend either workshop. The number of students in each possible group added together will sum to the total number of students in the class. Or we could subtract the known sizes of the groups from the total class size to determine the number of students who did not attend either workshop.

So, the number of students who did not attend either workshop is equal to the number of students in the class minus the number of students who attended both workshops minus the number of students attended only the “Know Your Birds” workshop minus the number of students who attended only the “Animals Around Us” workshop. Therefore, the number of students who did not attend either workshop is equal to $30 - 5 - 7 - 12 = 6$.

Therefore, 6 students missed out on the opportunity to learn from the workshop leaders.

Problem of the Week

Problem C

What's Not to Love?

At the start of the school year, students in Mr. Pi's class were asked the following question: "Do you love Math?" They were only allowed to answer "yes" or "no", and everyone had to answer. Of the 30 students in the class, 21 answered "yes" and 9 answered "no".

That day, with every student present, the probability of randomly selecting a student who answered the question "yes" was $\frac{21}{30} = \frac{7}{10}$ and the probability of randomly selecting a student who answered the question "no" was $\frac{9}{30} = \frac{3}{10}$.

However, on one particular morning later in the year, the following information was known about the class:

- at least one of the students who had answered "yes" was absent and at least one of the students who had answered "no" was absent;
- more than half of the class was present; and
- the probability of randomly selecting a student who had answered the question "yes" was $\frac{3}{4}$.

Is there enough information to determine how many students were absent that particular morning? If yes, how many students were absent? If no, explain why not.



Themes Data Management, Number Sense

Problem of the Week
Problem C and Solution
What's Not to Love?

Problem

At the start of the school year, students in Mr. Pi's class were asked the following question: "Do you love Math?" They were only allowed to answer "yes" or "no", and everyone had to answer. Of the 30 students in the class, 21 answered "yes" and 9 answered "no". So, with every student present, the probability of randomly selecting a student who answered the question "yes" was $\frac{21}{30} = \frac{7}{10}$ and the probability of randomly selecting a student who answered the question "no" was $\frac{9}{30} = \frac{3}{10}$.

However, on one particular morning later in the year, the following information was known about the class: at least one of the students who had answered "yes" was absent and at least one of the students who had answered "no" was absent; more than half of the class was present; and the probability of randomly selecting a student who had answered the question "yes" was $\frac{3}{4}$.

Is there enough information to determine how many students were absent that particular morning? If yes, how many students were absent? If no, explain why not.

Solution

Since at least one student from each of the two groups was absent, there were at least 2 students absent and at most 28 students present. Also, the maximum number of students who said "yes" would be $21 - 1 = 20$ and the maximum number of students who said "no" would be $9 - 1 = 8$. More than half the class was present so at least 16 students were present.

Since the probability of randomly selecting a student who answered "yes" was $\frac{3}{4}$, then the probability of randomly selecting a student who answered "no" was $\frac{1}{4}$.

We are looking for any number from 16 to 28 which is divisible by 4, so that when we find $\frac{3}{4}$ and $\frac{1}{4}$ of this number our result is a whole number. The numbers from 16 to 28 that are divisible by 4 are 16, 20, 24 and 28.

The following chart shows the results which are possible using the given information. There are 3 valid solutions that satisfy the given information. Therefore, there is not enough given information to determine the number of students who were absent that particular morning. The last solution in the chart in not valid. If the number present was 28, then 21 of those present would have answered the question "yes". But at least one student who answered "yes" was absent so the maximum number of students who answered "yes" would have been 20.

Number Present	Number Absent	Number who said "yes"	Number who said "no"	Valid / Not Valid
16	$30 - 16 = 14$	$\frac{3}{4} \times 16 = 12$	$\frac{1}{4} \times 16 = 4$	Valid
20	$30 - 20 = 10$	$\frac{3}{4} \times 20 = 15$	$\frac{1}{4} \times 20 = 5$	Valid
24	$30 - 24 = 6$	$\frac{3}{4} \times 24 = 18$	$\frac{1}{4} \times 24 = 6$	Valid
28	$30 - 28 = 2$	$\frac{3}{4} \times 28 = 21$	$\frac{1}{4} \times 28 = 7$	Not Valid

To Think About: Is there another piece of information that Mr. Pi could have provided so that two of the three valid answers could be eliminated leaving only one valid answer?

Computational Thinking (C)

TAKE ME TO THE COVER

Problem of the Week
Problem C
Burger Boxes

At the local burger restaurant, Patty always cooks burgers one at a time. After cooking a burger, she places it into one of three different boxes: one with dots, one with stripes, and one plain box. The first burger she places in a box with dots and puts it on the top of the stack. The second burger she places in a box with stripes and puts it on the top of the stack. The third burger she places in a plain box and puts it on the top of the stack. If she has cooked three burgers, she would have a stack as shown in Image 1.

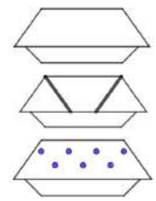


Image 1

Patty then repeats the pattern of placing the burgers in boxes. As she cooks a burger, she places that box on the top of the stack of not yet sold burgers, and continues to cycle through the three different boxes (dot, stripe, plain, dot, stripe, plain, ...) into which to place the burger. If she cooked two more burgers, she would have a stack as shown in Image 2.

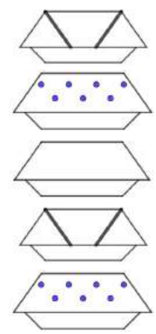


Image 2

At the same time, Tom sell burgers one at a time and always takes the uppermost box from the stack. Patty cooks faster than Tom can sell the burgers.

Shortly into a new shift, Patty has cooked some burgers and Tom has sold some burgers. The stack of unsold burgers looks like the stack in Image 3. What is the fewest number of burgers sold by Tom? Explain your answer.

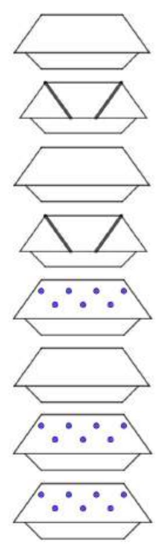


Image 3

Themes Algebra, Computational Thinking

Problem of the Week
Problem C and Solution
Burger Boxes

Problem

At the local burger restaurant, Patty always cooks burgers one at a time. After cooking a burger, she places it into one of three different boxes: one with dots, one with stripes, and one plain box. If she has cooked three burgers, she would have a stack as shown in Image 1. Patty then repeats the pattern of placing the burgers in boxes. As she cooks a burger, she places that box on the top of the stack of not yet sold burgers, and continues to cycle through the three different boxes (dot, stripe, plain, dot, stripe, plain, ...) into which to place the burger. If she cooked two more burgers, she would have a stack as shown in Image 2. At the same time, Tom sell burgers one at a time and always takes the uppermost box from the stack. Patty cooks faster than Tom can sell the burgers. Shortly into a new shift, Patty has cooked some burgers and Tom has sold some burgers. The stack of unsold burgers looks like the stack in Image 3. What is the fewest number of burgers sold by Tom? Explain your answer.

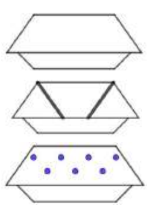


Image 1

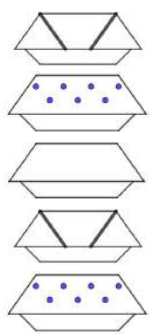


Image 2

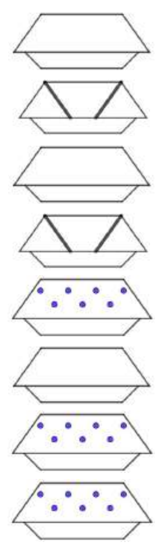


Image 3

Solution

Consider labelling the boxes as D (for dotted), S (for striped) and P (for plain, meaning no dots or stripes). So, we want to get a sequence that looks like: D, D, P, D, S, P, S, P where the leftmost boxes are at the bottom of the stack, and the rightmost boxes are at the top.

If no burgers were sold at all, we would have the sequence: $D, S, P, D, S, P, D, S, P, D, S, P, \ldots$

Notice that if we sell the burgers that are bolded as shown in the sequence:
$D, \mathbf{S}, \mathbf{P}, D, \mathbf{S}, P, D, S, P, \mathbf{D}, S, P,$ we have the sequence: D, D, P, D, S, P, S, P. This is the sequence that we are looking for. We can think of each bolded term as a burger being sold. The minimum number of boxes between the first two Ds is 2 (S, P). The minimum number of boxes between D and P is 1 (S). The minimum number boxes between the P and S is 1 (D). Therefore, the least amount of burgers sold is $2 + 1 + 1 = 4$.

The Beaver Computing Challenge (BCC):

This problem is based on a previous BCC problem. The BCC is designed to get students with little or no previous experience excited about computing. Questions are inspired by topics in computer science and connections to Computer Science are described in the solutions to all past BCC problems. If you enjoyed this problem, you may want to explore the BCC contest further.

Connections to Computer Science:

Computer scientists are concerned about how to efficiently store information. For certain problems, the best way to store information is in a data structure called a stack. A stack is a data structure that imposes the following rule about accessing data:

- new items can be put on the top of the stack (to become the new top of the stack): this is called pushing the element onto the stack
- items that are to be removed are removed from the top of the stack (making the element just below the top the new top of the stack): this is called popping the stack

Stacks are used for a variety of problem solving techniques, and perhaps the easiest one to visualize is the balanced parentheses problem. You would like to verify that some mathematical expression involving parentheses is valid. So (1+1), is valid, ((2+3)*(1+1)) is valid and so on. Ignoring any of the numbers or operators, we can ensure that we have a valid sequence of parentheses by the following simple process:

- read the mathematical expression from left-to-right;
- if we see (, push (onto the stack;
- if we see), pop the top (symbol from the stack;
- if we try to pop an empty stack, i.e., a stack without anything on it, the sequence is invalid;
- if we read the entire mathematical expression and the stack is not empty, the sequence is invalid;
- otherwise, the sequence is valid.

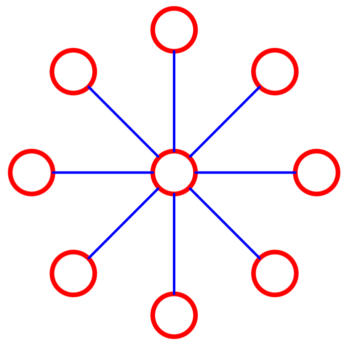
You can verify that the sequences above are verified by this algorithm, and that sequences like (((1+1) and)()()(would be correctly determined to be invalid by this algorithm.

Problem of the Week

Problem C

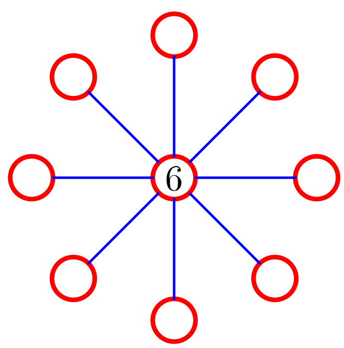
Fill to 15

The game, "Fill to 15", is a two-player game. The game board consists of 9 circles as shown in the following diagram.



The players alternate turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once in any game. The object of the game is to be the first player to place a disc so that the sum of the 3 numbers along a line through the centre circle is exactly 15.

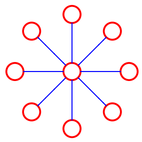
Alex and Blake play the game. Alex goes first. On her first move, Alex places a 6 in the centre circle. This is shown on the following diagram.



Then Blake places one of the eight remaining numbers in one of the empty circles on her first turn.

Show that no matter which numbers Blake plays, Alex can win the game on either her second or third turn.

Themes Computational Thinking, Number Sense



Problem of the Week

Problem C and Solution

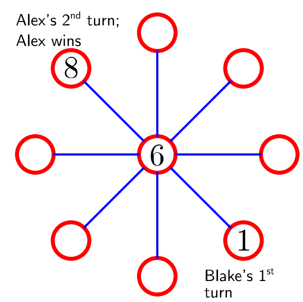
Fill to 15

Problem

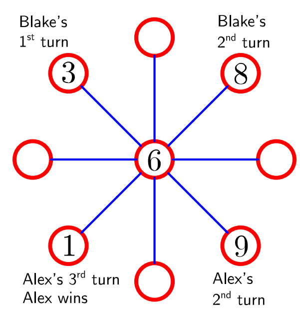
The game, "Fill to 15", is a two-player game. The game board consists of 9 circles as shown in the diagram above. The players alternate turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once in any game. The object of the game is to be the first player to place a disc so that the sum of the 3 numbers along a line through the centre circle is exactly 15. Alex and Blake play the game. Alex goes first. On her first move, Alex places a 6 in the centre circle. Then Blake places one of the eight remaining numbers in one of the empty circles on her first turn. Show that no matter which numbers Blake plays, Alex can win the game on either her second or third turn.

Solution

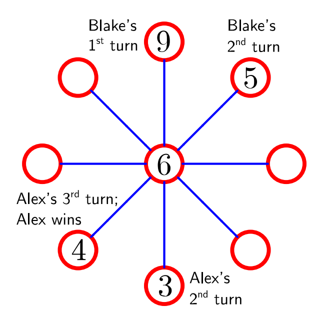
Since Alex played a 6 on her first turn, the other two discs in the line would need to add to 9 to make the total 15.



If, on her first turn, Blake plays one of the numbers 1, 2, 4, 5, 7 or 8, then there is an unused number that Alex can play on her second turn so that the sum of the line is 15. That is, if Blake plays a 1, then Alex will play an 8. (This is illustrated on the diagram to the right.) If Blake plays a 2, then Alex will play a 7. If Blake plays a 4, then Alex will play a 5. If Blake plays a 5, then Alex will play a 4. If Blake plays a 7, then Alex will play a 2. And if Blake plays an 8, then Alex will play a 1. In each of these 6 instances Alex can win on her second turn.



If, on her first turn, Blake places a 3 in any empty space, then the sum of the two discs in that line will be 9. Alex cannot win on her second turn since the only way to make the sum in that line 15 would be for her to play another 6. No number may be used more than once so this is not possible. However, if Alex completes the line by playing a 9 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7 and 8. Then, as we saw above, no matter what Blake plays on her second turn, there will be a number that Alex can place on that line so that the three numbers in the line add to 15. An example is illustrated in the diagram to the right.



Finally, if on her first turn, Blake places a 9 in any empty space, then the sum of the two discs on the line will already be 15 using just 2 discs. Alex cannot win on her second turn since playing any other disc in that line would make the sum greater than 15. However, if Alex completes the line by playing a 3 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7 and 8. Then, as we saw above, no matter what Blake plays next, there will be a number that Alex can place on that line so that the three numbers in the line add to 15. An example is illustrated in the diagram to the right.

Therefore, we have shown that no matter which numbers Blake plays, Alex can win the game on either her second or third turn.

Problem of the Week

Problem C

The Hat Race

Frankie, Chrystal, and Sam have just competed in the POTW's 10^{th} annual Toboggan Race.

Frankie, Chrystal, and Sam each finished in first, second, or third in the toboggan race. There were no ties.

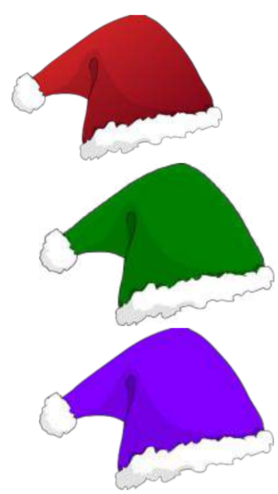
Each person also wore a different colour hat. One wore a red hat, one wore a green hat, and one wore a purple hat.

Using the following clues, determine who placed first, second and third, and which hat each person wore.

1. Chrystal was faster than Sam.
2. Sam did not wear the purple hat since she does not like purple and she did not finish before Frankie.
3. The person who wore the red hat was faster than the person who wore the green hat.
4. Chrystal did not wear the red hat and Frankie did not wear the green hat.
5. The person who came in first did not wear the purple hat.

You may find the following table a helpful way to organize your solution to this problem.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie						
Chrystal						
Sam						
First						
Second						
Third						



Theme Computational Thinking

Problem of the Week
Problem C and Solution
The Hat Race

Problem

Frankie, Chrystal, and Sam have just competed in the POTW's 10$^{\text{th}}$ annual Toboggan Race.

Frankie, Chrystal, and Sam each finished in first, second, or third in the toboggan race. There were no ties.

Each person also wore a different colour hat. One wore a red hat, one wore a green hat, and one wore a purple hat.

Using the following clues, determine who placed first, second and third, and which hat each person wore.

1. Chrystal was faster than Sam.
2. Sam did not wear the purple hat since she does not like purple and she did not finish before Frankie.
3. The person who wore the red hat was faster than the person who wore the green hat.
4. Chrystal did not wear the red hat and Frankie did not wear the green hat.
5. The person who came in first did not wear the purple hat.

Solution

Answer

We will present the answer first for those who want to check their work. The solution that follows represents one possible approach to arriving at a correct set of conclusions.

Frankie wore the red hat and finished in first.

Chrystal wore the purple hat and finished in second.

Sam wore the green hat and finished in third.

Solution

In our solution, we will we go through each clue and update the table based on the information in the clue. We will put an X in a cell if the combination indicated by the row and column for that cell is not possible, or a ✓ if it must be true.

From clue (1), since Chrystal was faster than Sam, we know that Sam could not have finished first and that Chrystal did not finish third. We can therefore put an X in the cells corresponding to Sam in first and Chrystal in third.

From clue (2), we can put an X in the cells corresponding to Sam wearing the purple hat and also to Frankie finishing in third since Frankie finished before Sam. The table is updated to the right.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie						X
Chrystal						X
Sam			X	X		
First						
Second						
Third						

From the previous table we see that neither Frankie nor Chrystal finished third. Therefore, Sam must have finished third. We can add a ✓ to the corresponding cell in the table.

Since Sam finished third, she could not have also finished in second. So, we can add an X to the corresponding cell in the table. The table is updated to the right.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie						X
Chrystal						X
Sam			X	X	X	✓
First						
Second						
Third						

From clue (3), we know that the person who wore the green hat did not finish first and that the person who wore the red hat did not finish third. We can therefore put an X in the appropriate cells.

Since we now know that Sam came in third, this clue also tells us that Sam is not the person who wore the red hat. We can put an X in the corresponding cell. The table is updated to the right.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie						X
Chrystal						X
Sam	X		X	X	X	✓
First		X				
Second						
Third	X					

From the previous table we see that Sam did not wear the red hat or the purple hat. Therefore, Sam must have worn the green hat.

Since Sam finished third, this also tells us that the person wearing the green hat finished third.

We can add a ✓ to the corresponding cells in the table. The table is updated to the right.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie						X
Chrystal						X
Sam	X	✓	X	X	X	✓
First		X				
Second						
Third	X	✓				

Since the people wear different hats, we now know that Chrystal and Frankie did not wear the green hat, and so we can add X's to the corresponding cells in the table.

Since we know that the person in the green hat finished third, they also could not have finished second. Similarly, the person in the purple hat could not have finished third. We can add X's to the corresponding cells in the table.

The table is updated to the right.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie		X				X
Chrystal		X				X
Sam	X	✓	X	X	X	✓
First		X				
Second		X				
Third	X	✓	X			

From clue (4), we can add an X to the cell corresponding to Chrystal wearing the red hat.

We know that neither Chrystal nor Sam wore the red hat. Therefore, Frankie must have worn the red hat. Similarly, Chrystal did not wear the red hat or the green hat. Therefore, Chrystal must have worn the purple hat.

We can add the corresponding ✓'s and X's. The table is updated to the right.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie	✓	X	X			X
Chrystal	X	X	✓			X
Sam	X	✓	X	X	X	✓
First		X				
Second		X				
Third	X	✓	X			

From clue (5), we know that the person who wore the purple hat did not come in first. Since Chrystal wore the purple hat, she did not come in first. This means that she must have finished second and it then follows that Frankie finished in first.

We can add the corresponding ✓'s and X's. The table is updated as shown to the right.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie	✓	X	X	✓	X	X
Chrystal	X	X	✓	X	✓	X
Sam	X	✓	X	X	X	✓
First		X				
Second		X				
Third	X	✓	X			

In the previous table the top three rows have been completed. We can use this information to fill in the four remaining empty cells in the bottom three rows of the table.

The completed table is shown to the right.

	Red Hat	Green Hat	Purple Hat	First	Second	Third
Frankie	✓	X	X	✓	X	X
Chrystal	X	X	✓	X	✓	X
Sam	X	✓	X	X	X	✓
First	✓	X	X			
Second	X	X	✓			
Third	X	✓	X			

From the completed table, we see that:

Frankie wore the red hat and finished in first.

Chrystal wore the purple hat and finished in second.

Sam wore the green hat and finished in third.

Problem of the Week
Problem C
Digit Swapping

Ali programs three buttons in a machine to swap some digits in a 4-digit number.

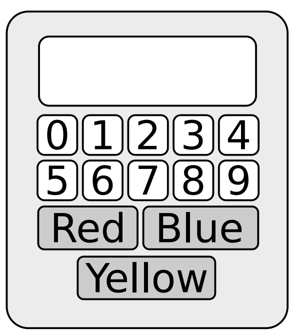
Red button:	swaps the thousands and tens digits
Blue button:	swaps the thousands and hundreds digits
Yellow button:	swaps the hundreds and units (ones) digits

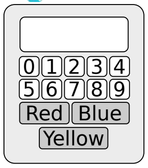
Ali types a 4-digit number into the machine. She then presses the following sequence of buttons to produce 6943 as the output.

Red | Yellow | Blue | Red | Yellow

What would the output have been if Ali had instead pressed the following sequence of buttons after typing in her original number?

Blue | Red | Yellow | Blue





Problem of the Week

Problem C and Solution

Digit Swapping

Problem

Ali programs three buttons in a machine to swap some digits in a 4-digit number.

Red button:	swaps the thousands and tens digits
Blue button:	swaps the thousands and hundreds digits
Yellow button:	swaps the hundreds and units (ones) digits

Ali types a 4-digit number into the machine. She then presses the following sequence of buttons to produce 6943 as the output.

Red | Yellow | Blue | Red | Yellow

What would the output have been if Ali had instead pressed the following sequence of buttons after typing in her original number?

Blue | Red | Yellow | Blue

Solution

First we need to know what number Ali initially typed into the machine in order to produce 6943 as the output.

???? → Red | Yellow | Blue | Red | Yellow → 6943

We can determine this by working backwards. This means we will start with 6943 and go through the sequence of buttons in the opposite order.

6943 → Yellow → 6349
6349 → Red → 4369
4369 → Blue → 3469
3469 → Yellow → 3964
3964 → Red → 6934

So Ali typed 6934 into the machine. Now we want to know the output after pressing the second sequence of buttons.

6934 → Blue | Red | Yellow | Blue → ????

We can go through the second button sequence to determine the new output.

6934 → Blue → 9634 → Red → 3694 → Yellow → 3496 → Blue → 4396

Therefore, the output would have been 4396.