



The **CENTRE** for **EDUCATION**
in **MATHEMATICS** and **COMPUTING**

Problem of the Week
Problems and Solutions
2019-2020

Problem E (Grade 11/12)

Themes

Number Sense (N)

Geometry (G)

Algebra (A)

Data Management (D)

Computational Thinking (C)

(Click on a theme name above to jump to that section)

*The problems in this booklet are organized into themes.
A problem often appears in multiple themes.

Number Sense (N)





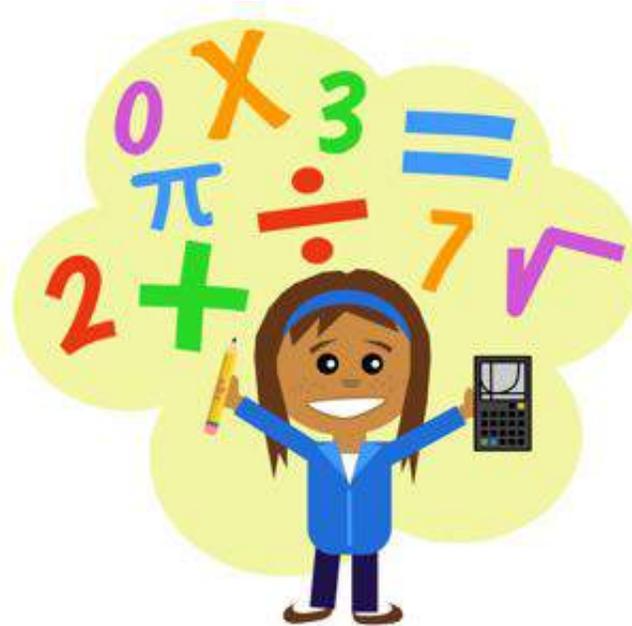
Problem of the Week

Problem E

Average Again

Erin has written three tests for her math class. She calculates the average of her marks on the first two tests. This average is then averaged with her third test mark to get 80%. She then calculates the average of her marks on the last two tests. This average is then averaged with her first test mark to get 83.5%. Finally, she calculates the average of her marks on the first and third tests. This average is then averaged with her second test mark to get 84.5%.

Her fourth test is coming up and after writing this test she wants her overall average for the four tests to be exactly 86%. If all four tests are out of 100 marks, what mark does Erin need to get on her fourth test in order for her overall average to be exactly 86%?





Problem of the Week

Problem E and Solution

Average Again

Problem

Erin has written three tests for her math class. She calculates the average of her marks on the first two tests. This average is then averaged with her third test mark to get 80%. She then calculates the average of her marks on the last two tests. This average is then averaged with her first test mark to get 83.5%. Finally, she calculates the average of her marks on the first and third tests. This average is then averaged with her second test mark to get 84.5%. Her fourth test is coming up and after writing this test she wants her overall average for the four tests to be exactly 86%. If all four tests are out of 100 marks, what mark does Erin need to get on her fourth test in order for her overall average to be exactly 86%?

Solution

Let a represent Erin's first test mark, b represent Erin's second test mark, c represent Erin's third test mark and d represent Erin's fourth test mark.

When the average of her first and second marks is averaged with her third test mark, the new average is 80 so $\frac{\frac{a+b}{2}+c}{2} = 80$. Multiplying by 2 gives $\frac{a+b}{2} + c = 160$. Multiplying by 2 again yields $a + b + 2c = 320$. (1)

When the average of her second and third marks is averaged with her first test mark, the new average is 83.5 so $\frac{\frac{b+c}{2}+a}{2} = 83.5$. Multiplying by 2 gives $\frac{b+c}{2} + a = 167$. Multiplying by 2 again yields $b + c + 2a = 334$. (2)

When the average of her first and third marks is averaged with her second test mark, the new average is 84.5 so $\frac{\frac{a+c}{2}+b}{2} = 84.5$. Multiplying by 2 gives $\frac{a+c}{2} + b = 169$. Multiplying by 2 again yields $a + c + 2b = 338$. (3)

By adding equations (1), (2) and (3) we obtain $4a + 4b + 4c = 992$, which simplifies to $a + b + c = 248$ after dividing by 4. This means that the sum of her first three marks is 248.

To obtain an average of 86% over the four tests, Erin needs a total of $86 \times 4 = 344$ marks. In other words, $a + b + c + d = 344$. We know that $a + b + c = 248$ so $248 + d = 344$, and $d = 96$ follows.

Therefore, to obtain an average of 86%, Erin needs 96% on her fourth test.

It would be a straight forward process to determine Erin's first three test marks but our question did not ask us to do this. However, for the curious, on her first test Erin got 86, on her second test Erin got 90, and on her third test Erin got 72.



Problem of the Week

Problem E

Definitely Not Odd

Timmy creates nine-digit positive integers by using the digits from 1 to 9 each exactly once. How many of these nine-digit integers are a multiple of 4 or have a units digit that is an 8?

Number	Divisible by 4	Units Digit is 8
987612345	X	X
123456798	X	✓
198764352	✓	X
937614528	✓	✓





Number	Divisible by 4	Units Digit is 8
987612345	X	X
123456798	X	✓
198764352	✓	X
937614528	✓	✓

Problem of the Week

Problem E and Solution

Definitely Not Odd

Problem

Timmy creates nine-digit positive integers by using the digits from 1 to 9 each exactly once. How many of these nine-digit integers are a multiple of 4 or have a units digit that is an 8?

Solution

In creating a 9-digit positive integer with distinct digits from 1 to 9, there are 9 choices for the first digit, 8 choices for the second digit, 7 choices for the third digit, and so on. So the number of 9-digit numbers with distinct digits is $9 \times 8 \times 7 \times \cdots \times 2 \times 1 = 9! = 362\,880$.

First, we will count the number of 9-digit positive integers with distinct digits from 1 to 9 that end in an 8. If the 9-digit integer ends in 8, there is only choice for the unit's digit. There are then $8!$ ways to create the remaining 8-digit number. Thus, the number of 9-digit positive integers with distinct digits from 1 to 9 that end in 8 is equal to $1 \times 8! = 40\,320$.

Next, we will count the number of 9-digit positive integers with distinct digits from 1 to 9 that are divisible by 4. For a number to be divisible by 4, the last two digits of the number must be divisible by 4. There are 24 positive integers less than 100 that are divisible by 4. This group includes two 1-digit numbers, 4 and 8, which must be excluded. It also includes four 2-digit numbers ending in zero, 20, 40, 60 and 80. Since zero is not one of the nine possible digits, these numbers must be excluded. And finally, the list includes two numbers, 44 and 88, which have repeated digits. Since each digit can be used only once, these numbers must be excluded. Therefore, there are $24 - 2 - 4 - 2 = 16$ valid 2-digit numbers which are divisible by 4. There are then $7!$ ways to create the remaining 7-digit number. Therefore, the number of 9-digit positive integers with distinct digits from 1 to 9 that are divisible by 4 is equal to $16 \times 7! = 80\,640$.

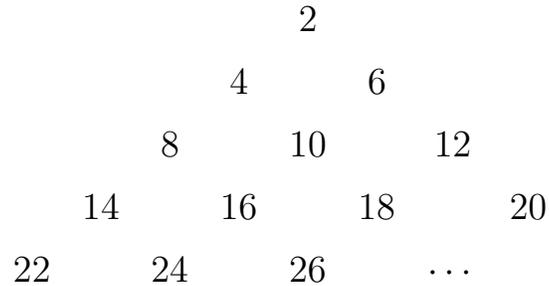
If we added the $1 \times 8!$ and the $16 \times 7!$ we would be double counting the integers that end in 8 **and** are divisible by 4. So, we must subtract off the number of integers which are divisible by 4 and end in 8 since they have been counted twice. There are three 2-digit numbers with distinct digits that end in 8 and are divisible by 4. They are 28, 48 and 68. As before, there are $7!$ ways to create the remaining 7-digit number. So the number of 9-digit positive integers with distinct digits from 1 to 9 ending in 8 **and** divisible by 4 is equal to $3 \times 7! = 15\,120$.

Therefore, the number of 9-digit integers that have distinct digits from 1 to 9 and are a multiple of 4 or have a units digit that is an 8 is equal to $40\,320 + 80\,640 - 15\,120 = 105\,840$.



Problem of the Week
Problem E
This is the Year 2

The positive even integers are arranged in increasing order in a triangle, as shown:



Each row contains one more number than the previous row. One of the rows of the triangle contains the number 2020. Determine the sum of the numbers in the row that contains 2020.



Did you know?

There is a quick way to calculate the sum $1 + 2 + 3 + 4 + \dots + 19 + 20$?

$$1 + 2 + 3 + 4 + \dots + 19 + 20 = \frac{(20)(20 + 1)}{2} = 210$$

In general, it can be shown that if n is a positive integer, then the sum of the integers from 1 to n is $S = 1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$.





Problem of the Week

Problem E and Solution

This is the Year 2

Problem

The positive even integers are arranged in increasing order in a triangle, as shown:

			2			
		4		6		
	8		10		12	
14		16		18		20
22	24		26		...	

Each row contains one more number than the previous row. One of the rows of the triangle contains the number 2020. Determine the sum of the numbers in the row that contains 2020.

Solution

To begin the solution, we make an observation: the row number in the triangle is the same as the number of numbers in that particular row. For example, row 1 contains 1 number, row 2 contains 2 numbers, and row r contains r numbers. If we add up all of the row numbers from row 1 to row r we get the number of numbers in the table and the largest number in the row is $2r$.

For example, if we calculate the sum $1 + 2 + 3 + 4$, we get 10. This number corresponds to the number of numbers in the first four rows of the triangle and the largest number in the fourth row is $2 \times 10 = 20$.

Let n represent the row number of the row that contains the number 2020. Since $2020 = 2 \times 1010$ we need to find the row where the 1010th number occurs.

Using the formula for the sum of the integers from 1 to n , we want $\frac{n(n+1)}{2} \geq 1010$. Multiplying by 2, the equation simplifies to $n(n+1) \geq 2020$.

We will use trial and error to find the possible value of n . Finding the $\sqrt{2020}$ will get us a good place to start checking values of n . ($\sqrt{2020} \approx 44.9$.)

When $n = 44$, $44 \times 45 = 1980 < 2020$ and $\frac{44 \times 45}{2} = 990$.

When $n = 45$, $45 \times 46 = 2070 > 2020$ and $\frac{45 \times 46}{2} = 1035$.

These guesses are quite useful. When there are 44 rows in the triangle, the last number in the 44th row is $990 \times 2 = 1980$. So the first number in the 45th row is 1982. When there are 45 rows in the triangle, the last number in the 45th row is $2 \times 1035 = 2070$. We want the sum of the numbers in the 45th row. To do this we will add all of the even integers from 2 to 2020. But this sum includes extra even integers from 2 to 1980. We will use our formula to compute this second sum and subtract it from the first sum to obtain the desired result.



$$\begin{aligned}\text{sum} &= 1982 + 1984 + 1986 + \cdots + 2066 + 2068 + 2070 \\ &= (\mathbf{2 + 4 + 6 + \cdots + 1976 + 1978 + 1980} + 1982 + 1984 + 1986 + \cdots + 2066 + 2068 + 2070) \\ &\quad - (\mathbf{2 + 4 + 6 + \cdots + 1976 + 1978 + 1980}) \\ &= (2(\mathbf{1 + 2 + 3 + \cdots + 988 + 989 + 990} + 991 + 992 + 993 + \cdots + 1033 + 1034 + 1035)) \\ &\quad - (2(\mathbf{1 + 2 + 3 + \cdots + 988 + 989 + 990})) \\ &= 2 \times \left(\frac{1035 \times 1036}{2} \right) - 2 \times \left(\frac{990 \times 991}{2} \right) \\ &= 91\,170\end{aligned}$$

The sum of the numbers in the row containing 2020 is 91 170.



Problem of the Week

Problem E

Sum Product Equality

The number 8 is the sum and product of the numbers in the collection of four positive integers $(1, 1, 2, 4)$, since $1 + 1 + 2 + 4 = 8$ and $1 \times 1 \times 2 \times 4 = 8$.

The number 2020 can also be made from a collection of n positive integers, with $n > 1$, that multiply to 2020 and add to 2020.

Determine all possible values for n .

$$+ = \times$$





$$+ = \times$$

Problem of the Week

Problem E and Solution

Sum Product Equality

Problem

The number 8 is the sum and product of the numbers in the collection of four positive integers $(1, 1, 2, 4)$, since $1 + 1 + 2 + 4 = 8$ and $1 \times 1 \times 2 \times 4 = 8$. The number 2020 can be made from a collection of n positive integers, with $n > 1$, that multiply to 2020 and add to 2020. Determine all possible values for n .

Solution

To form such a collection of integers, our strategy is to determine all collections of integers larger than 1 whose product is 2020, and then for each collection add enough 1s to make the sum of the numbers in the collection 2020.

Since we want to consider integers whose product is 2020, we should find the divisors of 2020. The prime factorization of 2020 is $2020 = 2 \times 2 \times 5 \times 101$. We can use this factorization to determine all collections of integers greater than 1 that multiply to 2020. The only collection with 4 numbers is $(2, 2, 5, 101)$. The collections with 3 numbers are $(4, 5, 101)$, $(2, 10, 101)$, $(2, 5, 202)$, and $(2, 2, 505)$. A systematic count shows that the only collections with 2 numbers are $(4, 505)$, $(5, 404)$, $(10, 202)$, $(2, 1010)$, and $(20, 101)$. It is left for the reader to verify this. We will not consider the collection (2020) , as it is given that the number of integers in the collection is to be greater than 1.

We show these collections in the first column of the table below. We also determine the sum of the integers in this collection, the number of 1s needed (subtract the sum from 2020), and the value of n , which is equal to the number of integers in first column plus the entry in third column.

Collection of integers greater than 1	Sum of integers greater than 1 in collection	Number of 1s needed to sum to 2020	Value of n
$(2, 2, 5, 101)$	110	1910	$1910 + 4 = 1914$
$(4, 5, 101)$	110	1910	$1910 + 3 = 1913$
$(2, 10, 101)$	113	1907	$1907 + 3 = 1910$
$(2, 5, 202)$	209	1811	$1811 + 3 = 1814$
$(2, 2, 505)$	509	1511	$1511 + 3 = 1514$
$(4, 505)$	509	1511	$1511 + 2 = 1513$
$(5, 404)$	409	1611	$1611 + 2 = 1613$
$(10, 202)$	212	1808	$1808 + 2 = 1810$
$(2, 1010)$	1012	1008	$1008 + 2 = 1010$
$(20, 101)$	121	1899	$1899 + 2 = 1901$

Therefore, the possible values for n are 1010, 1513, 1514, 1613, 1810, 1814, 1901, 1910, 1913, and 1914.



Problem of the Week

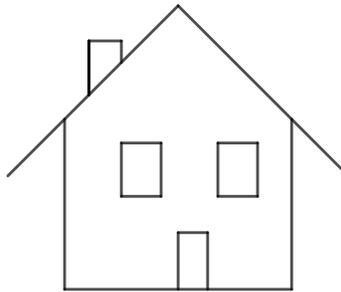
Problem E

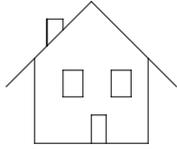
Square Neighbours

Alyssa, Bilal, Constantine and Daiyu all live on the same street.

A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Constantine's house number is 31 more than Alyssa's house number and Daiyu's house number is 31 more than Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number.

What is the house number of each person?





Problem of the Week

Problem E and Solution

Square Neighbours

Problem

Alyssa, Bilal, Constantine and Daiyu all live on the same street. A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Constantine's house number is 31 more than Alyssa's house number and Daiyu's house number is 31 more than Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number. What is the house number of each person?

Solution

Both Alyssa's house number and Bilal's house number must be two-digit numbers. If Alyssa's house number is a one-digit number, Bilal's house number would have to be a three-digit number to create the four-digit perfect square. This means that Constantine's house number would be a two-digit number and Daiyu's house number would be at least a three-digit number and when you combine these two numbers it will be at least a five-digit number. A similar argument can be presented if Bilal's house number is a one-digit number. Therefore, both Alyssa and Bilal have house numbers that are each two-digit numbers.

Let Alyssa's house number be x and Bilal's house number be y . Then $100x + y$ is the four digit number created by writing Alyssa's house number followed by Bilal's house number. But $100x + y$ is a perfect square, so let $100x + y = k^2$, for some positive integer k . (1)

Therefore, Constantine's house number is $(x + 31)$ and Daiyu's house number is $(y + 31)$. The new number created by writing Constantine's house number followed by Daiyu's house number is $100(x + 31) + (y + 31)$. This new four-digit number is also a perfect square. So $100(x + 31) + (y + 31) = m^2$, for some positive integer m , with $m > k$. This simplifies as follows:

$$\begin{aligned} 100x + 3100 + y + 31 &= m^2 \\ 100x + y + 3131 &= m^2 \end{aligned} \quad (2)$$



From (1) above, we have $100x + y = k^2$. Substituting k^2 for $100x + y$ in (2) we obtain $k^2 + 3131 = m^2$ or $3131 = m^2 - k^2$. We observe that $m^2 - k^2$ is a difference of squares, so $m^2 - k^2 = (m + k)(m - k) = 3131$.

Since m and k are positive integers, $m + k$ is a positive integer and $m + k > m - k$. Also, $m - k$ must be a positive integer since $(m + k)(m - k) = 3131$. So we are looking for two positive integers that multiply to 3131. There are two possibilities, 3131×1 or 101×31 .

First we will examine $(m + k)(m - k) = 3131 \times 1$. From this we obtain two equations in two unknowns, namely $m + k = 3131$ and $m - k = 1$. Subtracting the two equations gives $2k = 3130$ or $k = 1565$. Then $k^2 = 1565^2 = 2\,449\,225$. This is not a four-digit number, so 3131×1 is not an admissible factorization of 3131.

Next we examine $(m + k)(m - k) = 101 \times 31$. This leads to $m + k = 101$ and $m - k = 31$. Subtracting the two equations we get $2k = 70$ or $k = 35$. Then $100x + y = k^2 = 1225$. Therefore, $x = 12$ and $y = 25$, since 1225 is the four-digit number formed by writing Alyssa's house number, x , followed by Bilal's house number, y .

Now, Constantine's house number $x + 31 = 12 + 31 = 43$ and Daiyu's house number is $y + 31 = 25 + 31 = 56$. Notice that $4356 = 66^2$, so it is a four-digit perfect square.

Therefore, Alyssa's house number is 12, Bilal's house number is 25, Constantine's house number is 43, and Daiyu's house number is 56.



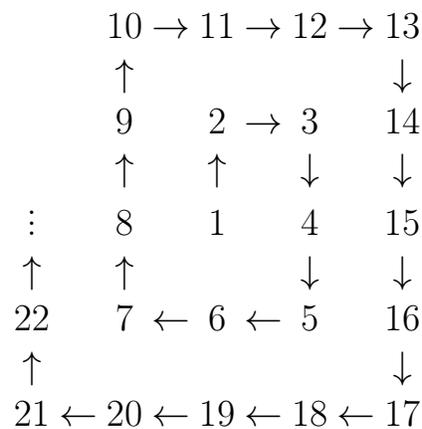
Problem of the Week

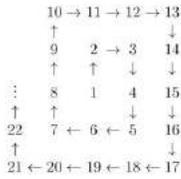
Problem E

Spiral

A spiral of numbers is created, as shown, starting with 1.

If the pattern of the spiral continues, how will the numbers 2020, 2021, and 2022 appear in the spiral? (Will they appear left to right in a row? Right to left in a row? Down in a column? Up in a column? Down and then left? Up and then right?)





Problem of the Week

Problem E and Solution

Spiral

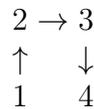
Problem

A spiral of numbers is created, as shown above, starting with 1. If the pattern of the spiral continues, how will the numbers 2020, 2021, and 2022 appear in the spiral? (Will they appear left to right in a row? Right to left in a row? Down in a column? Up in a column? Down and then left? Up and then right?)

Solution

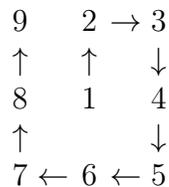
We are looking for a pattern which can be used to predict the positions of the numbers 2020, 2021, and 2022.

Observe a first “square” in the middle containing the numbers 1, 2, 3, and 4, with 4 in the lower right corner.



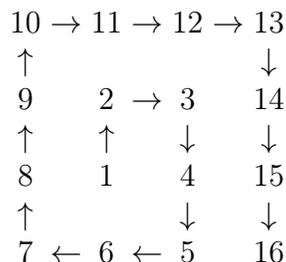
Starting with the 1, to get this “square” we need to add one number above and one number to right. We also need to add a number that is in the diagonal up and to the right. That is, we need to add three numbers to get the first “square”.

Extend the diagram to create a second “square”:



To get this second “square”, we need to add two numbers below the first “square” and two numbers to the left of the first “square”. We also need to add a number that is in the diagonal down and to the left. That is, we need to add five numbers to get from the first “square” to the second “square”.

Extend the diagram to create a third “square”:

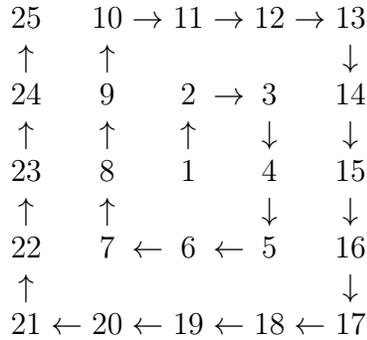


To get this third “square”, we need to add three numbers above the second “square” and three numbers to the right of the second “square”. We also need to add a number that is in the



diagonal up and to the right. That is, we need to add seven numbers to get from the second “square” to the third “square”.

Extend the diagram to create a fourth “square”:



To get this fourth “square” we need to add four numbers below the third “square” and four numbers to the left of the third “square”. We also need to add a number that is in the diagonal down and to the left. That is, we need to add nine numbers to get from the third “square” to the fourth “square”.

If this pattern continues, the fifth “square” would ‘end’ with $25 + 11 = 36$ in the bottom right position. The sixth “square” would end with $36 + 13 = 49$ in the top left. Notice that the numbers 4, 9, 16, 25, 36, and 49 are all perfect squares. Also notice that the even perfect squares are in the bottom right corners and the odd perfect squares are in the top left corners. Now, 2020 lies between $44^2 = 1936$ and $45^2 = 2025$. Since 2025 is an odd perfect square, 2025 will be in the top left corner of a “square”, and the left side of the “square” will look like this:



Therefore, 2020, 2021 and 2022 will be going up in a column.



Problem of the Week

Problem E

Uphill then Down

Two racers compete in a 1400 m race. The first 700 m of the race is uphill, and the second 700 m is down the same hill along the same path. Each racer's constant uphill speed is half of their respective constant downhill speed.

The faster racer reaches the top of the hill and immediately starts running downhill. The faster racer then meets the slower racer (still going uphill) 70 m from the top of the hill.

When the faster racer finishes the race, how far is the slower racer from the finish line?





Problem of the Week

Problem E and Solution

Uphill then Down

Problem

Two racers compete in a 1400 m race. The first 700 m of the race is uphill, and the second 700 m is down the same hill along the same path. Each racer's constant uphill speed is half of their respective constant downhill speed.

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When the faster racer finishes the race, how far is the slower racer from the finish line?

Solution

Let a represent the faster racer's uphill speed and $2a$ represent their downhill speed.

Let b represent the slower racer's uphill speed and $2b$ represent their downhill speed.

We know that *distance* = *speed* \times *time* or *time* = *distance* \div *speed*.

When the faster racer and the slower racer meet 70 m from the top of the hill, their times are the same. At this time the slower racer has run $700 - 70 = 630$ m uphill. Therefore:

$$\text{faster's time up 700 m} + \text{faster's time down 70 m} = \text{slower's time up 630 m}$$

$$\frac{700}{a} + \frac{70}{2a} = \frac{630}{b}$$

Reducing the second fraction:
$$\frac{700}{a} + \frac{35}{a} = \frac{630}{b}$$

Simplifying:
$$\frac{735}{a} = \frac{630}{b}$$

By rearranging, we obtain
$$\frac{a}{b} = \frac{735}{630} = \frac{7}{6}. \quad (1)$$

Solution continues on the next page.



Let x represent the distance the slower racer is behind the faster racer at the end of the race.

When the faster racer reaches the finish, they have travelled 700 m uphill at a speed of a and 700 m down at a speed of $2a$. The slower racer has travelled 700 m uphill at a speed of b and $(700 - x)$ m down at a speed of $2b$.

When the faster racer reaches the finish line and the slower racer is x m from the finish, their times are the same. Therefore,

$$\begin{aligned}\frac{700}{a} + \frac{700}{2a} &= \frac{700}{b} + \frac{700 - x}{2b} \\ \frac{1400}{2a} + \frac{700}{2a} &= \frac{1400}{2b} + \frac{700 - x}{2b} \\ \frac{2100}{2a} &= \frac{2100 - x}{2b}\end{aligned}$$

Multiplying both sides by by 2,

$$\frac{2100}{a} = \frac{2100 - x}{b}$$

It follows that $\frac{a}{b} = \frac{2100}{2100 - x}$. But from (1) earlier, $\frac{a}{b} = \frac{7}{6}$.

$$\text{Therefore, } \frac{2100}{2100 - x} = \frac{7}{6}$$

$$\begin{aligned}\text{Solving for } x, \quad 12\,600 &= 14\,700 - 7x \\ 7x &= 2100 \\ x &= 300\end{aligned}$$

Therefore, the slower racer is 300 m from the finish line when the faster racer finishes the race.



Problem of the Week

Problem E

What's Your Unlucky Number?

Everyone has a lucky number. Sue Perstitious does not have a lucky number but considers the number 13 to be unlucky.

Three bags each contain tokens. The green bag contains 20 round green tokens, each with a different integer from 1 to 20. The red bag contains 12 triangular red tokens, each with a different integer from 1 to 12. The blue bag contains 8 square blue tokens, each with a different integer from 1 to 8.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $20 \times 12 \times 8 = 1920$ different combinations of tokens that can be created by selecting one token from each bag. Note that the order of selection does not matter. Also note that selecting the 7 red token, the 5 blue token and 1 green token is different than selecting the 5 red token, 7 blue token and the 1 green token.



Sue selects one token from each bag. What is the probability that the sum of the numbers selected is divisible by 13, her unlucky number?



Problem of the Week

Problem E and Solution

What's Your Unlucky Number?

Problem

Everyone has a lucky number. Sue Perstitious does not have a lucky number but considers the number 13 to be unlucky. Three bags each contain tokens. The green bag contains 20 round green tokens, each with a different integer from 1 to 20. The red bag contains 12 triangular red tokens, each with a different integer from 1 to 12. The blue bag contains 8 square blue tokens, each with a different integer from 1 to 8.

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Sue selects one token from each bag. What is the probability that the sum of the numbers selected is divisible by 13, her unlucky number?

Solution

There are 20 different numbers which can be selected from the green bag, 12 different numbers which can be selected from the red bag, and 8 different numbers which can be selected from the blue bag. So there are $20 \times 12 \times 8 = 1920$ different combinations of numbers which can be selected from the three bags.

Let (g, r, b) represent the outcome of a selection where g is the number on the token selected from the green bag, r is the number on the token selected from the red bag and b is the number on the token selected from the blue bag. Also, $1 \leq g \leq 20$, $1 \leq r \leq 12$, and $1 \leq b \leq 8$, for integers g, r, b .

Numbers that are divisible by 13 are 13, 26, 39, 52, \dots . The maximum sum that can be reached any selection is $8 + 12 + 20 = 40$. To count the number of possibilities for sums which are divisible by 13, we will consider three cases: a sum of 13, a sum of 26 and a sum of 39. Within the first two cases, we will look at sub-cases based on the possible outcome for the 8 possible selections from the blue bag.

1. The sum of the numbers on the 3 tokens is 13.

• 1 is on the token selected from the blue bag

The sum of the numbers on the other two tokens is 12. Selecting the 12 token from the red bag is not possible since the number on the token selected from the green bag must be at least 1. The possibilities for (g, r, b) are $(1, 11, 1), (2, 10, 1), (3, 9, 1), \dots, (11, 1, 1)$, 11 possibilities in total.

• 2 is on the token selected from the blue bag

The sum of the numbers on the other two tokens is 11. Selecting the 11 or 12 token from the red bag is not possible since the number on the token selected from the green bag must be at least 1. The possibilities for (g, r, b) are $(1, 10, 2), (2, 9, 2), (3, 8, 2), \dots, (10, 1, 2)$, 10 possibilities in total.



- **3 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 10. Using reasoning similar to the preceding two cases, the possibilities for (g, r, b) are $(1, 9, 3), (2, 8, 3), (3, 7, 3), \dots, (9, 1, 3)$, 9 possibilities in total.

- **4 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 9. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 8, 4), (2, 7, 4), (3, 6, 4), \dots, (8, 1, 4)$, 8 possibilities in total.

- **5 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 8. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 7, 5), (2, 6, 5), (3, 5, 5), \dots, (7, 1, 5)$, 7 possibilities in total.

- **6 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 7. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 6, 6), (2, 5, 6), (3, 4, 6), \dots, (6, 1, 6)$, 6 possibilities in total.

- **7 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 6. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 5, 7), (2, 4, 7), (3, 3, 7), (4, 2, 7), (5, 1, 7)$, 5 possibilities in total.

- **8 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 5. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 4, 8), (2, 3, 8), (3, 2, 8), (4, 1, 8)$, 4 possibilities in total.

Summing the results from the 8 cases, there are $11 + 10 + 9 + \dots + 5 + 4 = 60$ combinations so that the sum of numbers on the 3 tokens is 13.

2. The sum of the numbers on the 3 tokens is 26.

- **1 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 25. The largest number possible from the green bag is 20 so the smallest possible number from the red bag would be 5. The largest number possible from the red bag is 12 so the smallest possible number from the green bag would be 13. The numbers from the green bag go from 20 to 13 while the numbers from the red bag go from 5 to 12. The possibilities for (g, r, b) are $(20, 5, 1), (19, 6, 1), (18, 7, 1), \dots, (13, 12, 1)$, 8 possibilities in total.

- **2 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 24. Using similar reasoning to the first case in this section, the possibilities for (g, r, b) are $(20, 4, 2), (19, 5, 2), (18, 6, 2), \dots, (12, 12, 2)$, 9 possibilities in total.

- **3 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 23. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are $(20, 3, 3), (19, 4, 3), (18, 5, 3), \dots, (11, 12, 3)$, 10 possibilities in total.



- **4 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 22. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are $(20, 2, 4), (19, 3, 4), (18, 4, 4), \dots, (10, 12, 4)$, 11 possibilities in total.

- **5 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 21. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are $(20, 1, 5), (19, 2, 5), (18, 3, 5), \dots, (9, 12, 5)$, 12 possibilities in total.

- **6 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 20. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 8. The highest number possible from the green bag is 19 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are $(19, 1, 6), (18, 2, 6), (17, 3, 6), \dots, (8, 12, 6)$, 12 possibilities in total.

- **7 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 19. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 7. The highest number possible from the green bag is 18 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are $(18, 1, 7), (17, 2, 7), (16, 3, 7), \dots, (7, 12, 7)$, 12 possibilities in total.

- **8 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 18. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 6. The highest number possible from the green bag is 17 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are $(17, 1, 8), (16, 2, 8), (15, 3, 8), \dots, (6, 12, 8)$, 12 possibilities in total.

Summing the results from the cases, there are $8 + 9 + 10 + 11 + 4(12) = 86$ combinations so that the sum of the numbers on the 3 tokens is 26.

3. The sum of the numbers on the 3 tokens is 39.

The maximum sum that can be obtained is $8 + 12 + 20 = 40$. A sum of 39 can only be achieved by keeping two of the three tokens at their maximum and reducing the third token to 1 less than its maximum. The possibilities for (g, r, b) are $(20, 12, 7), (20, 11, 8)$ and $(19, 12, 8)$, 3 possibilities in total.

The total number of combinations in which the sum of the numbers on the three tokens is divisible by 13 is $60 + 86 + 3 = 149$.

Therefore, the probability of selecting three tokens with a sum which is divisible by 13 is $\frac{149}{1920}$.

There is less than an 8% chance that the three tokens selected by Sue will sum to a multiple of her unlucky number.



Problem of the Week

Problem E

CUBES

Certain numbers have interesting properties. For example, $1^3 + 5^3 + 3^3 = 153$. That is, the sum of the cubes of the individual digits of the positive integer 153 is the number itself. This may lead you to ask a question like, “Are there other such numbers?” (Yes there are, but that is not our concern today.)

The number 512 stands alone as a three-digit positive integer with three different digits such that the cube of the sum of the digits equals the number itself. That is, $(5 + 1 + 2)^3 = 512$. This is the only three-digit positive integer with three distinct digits that has this property.

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number.

That is, find all five-digit positive integers of the form *CUBES* with distinct digits such that

$$(C + U + B + E + S)^3 = CUBES$$



Problem of the Week

$(C+U+B+E+S)^3 = CUBES$ Problem E and Solution

CUBES

Problem

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number. That is, find all five-digit positive integers of the form $CUBES$ with distinct digits such that $(C + U + B + E + S)^3 = CUBES$.

Solution

A straightforward approach to solving this problem is to determine the smallest possible number and the largest possible number. Then, work at finding the numbers in that range that satisfies the given property.

The smallest five-digit number with distinct digits is 10 234. Since $\sqrt[3]{10\,234} \doteq 21.7$, the smallest number to consider is $22^3 = 10\,648$. The largest sum of five distinct digits is $9 + 8 + 7 + 6 + 5 = 35$, so the largest number to consider is $35^3 = 42\,875$. The possibilities, if any exist, are from 22^3 to 35^3 . We need to examine these cubes to find the solution.

Number	Number ³	Sum of the Digits	Has the Property?
22	10 648	19	no, $22 \neq 19$
23	12 167		no, digits not distinct
24	13 824	18	no, $24 \neq 18$
25	15 625		no, digits not distinct
26	17 576		no, digits not distinct
27	19 683	27	Yes, $(1 + 9 + 6 + 8 + 3)^3 = 19683$
28	21 952		no, digits not distinct
29	24 389	26	no, $29 \neq 26$
30	27 000		no, digits not distinct
31	29 791		no, digits not distinct
32	32 768	26	no, $32 \neq 26$
33	35 937		no, digits not distinct
34	39 304		no, digits not distinct
35	42 875	26	no, $35 \neq 26$

Since we have examined all possibilities, we can conclude that 19 683 is the only five-digit positive integer with distinct digits such that the cube of the sum of the digits of the number equals the original number.



Problem of the Week

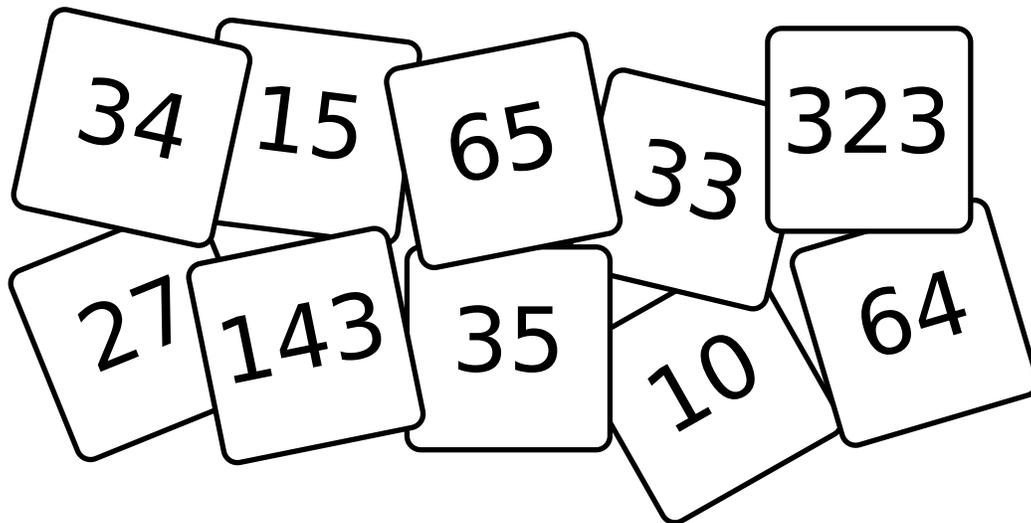
Problem E

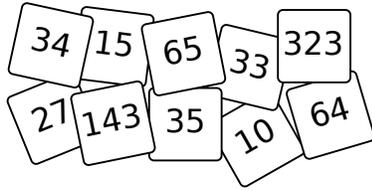
The Factor Flip

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up.

Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up.

List all the possible orders in which Dani can flip the cards so that all cards get flipped over.





Problem of the Week

Problem E and Solution

The Factor Flip

Problem

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up. Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up. List all the possible orders in which Dani can flip the cards so that all cards get flipped over.

Solution

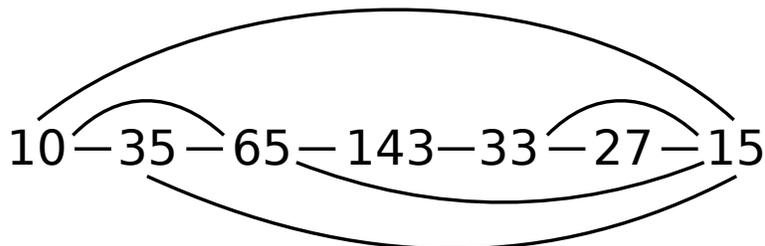
We will start by writing down the prime factors for each of the numbers on the cards.

Number	Prime Factors	Number	Prime Factors
10	2,5	35	5,7
15	3,5	64	2
27	3	65	5,13
33	3,11	143	11,13
34	2,17	323	17,19

Notice the number 323 shares a prime factor with only the number 34. That means we must start (or end) with 323.

If we start with 323, then the next number must be 34. From 34, the next number could be 64 or 10. If the next number were 10, then in order to eventually flip over the 64 card, the 64 must follow the 10, and at this point no more cards can be flipped. Therefore, in order to flip all the cards, the first four numbers flipped must be 323, 34, 64, and then 10.

From this point, we can draw lines between numbers that share prime factors to create the following diagram.





We now just need to figure out the number of paths through the diagram, starting at 10 that use each number exactly once.

After 10, the next number can be 35, 65, or 15. In each case, we carefully trace through all possible paths.

Case 1: The number 35 follows the number 10. That gives us the following five possible paths.

35, 15, 65, 143, 33, 27

35, 15, 27, 33, 143, 65

35, 65, 15, 27, 33, 143

35, 65, 143, 33, 27, 15

35, 65, 143, 33, 15, 27

Case 2: The number 65 follows the number 10. That gives us the following two possible paths.

65, 143, 33, 27, 15, 35

65, 35, 15, 27, 33, 143

Case 3: The number 15 follows the number 10. That gives us the following two possible paths.

15, 27, 33, 143, 65, 35

15, 35, 65, 143, 33, 27

Starting with the card numbered 323, we have found that there is a total of nine orders for flipping the cards:

323, 34, 64, 10, 35, 15, 65, 143, 33, 27

323, 34, 64, 10, 35, 15, 27, 33, 143, 65

323, 34, 64, 10, 35, 65, 15, 27, 33, 143

323, 34, 64, 10, 35, 65, 143, 33, 27, 15

323, 34, 64, 10, 35, 65, 143, 33, 15, 27

323, 34, 64, 10, 65, 143, 33, 27, 15, 35

323, 34, 64, 10, 65, 35, 15, 27, 33, 143

323, 34, 64, 10, 15, 27, 33, 143, 65, 35

323, 34, 64, 10, 15, 35, 65, 143, 33, 27

Each of these can be reversed, so the total number of possible orders is 18.

Therefore, there are 18 orders in which Dani can flip the cards so that all cards get flipped over. They are the 9 orders listed above and the reverse of each.

Geometry (G)





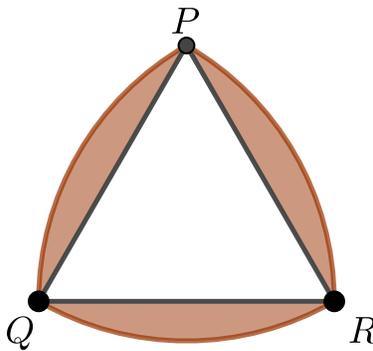
Problem of the Week

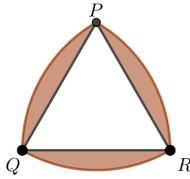
Problem E

What's Outside?

In the diagram below, $\triangle PQR$ is an equilateral triangle with sides of length 2 cm. Arc PQ is an arc of a circle with centre R and radius RQ . Arc PR and arc RQ are similarly drawn with centres Q and P , respectively.

Determine the total area of the shaded regions in the diagram.





Problem of the Week

Problem E and Solution

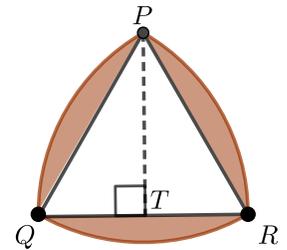
What's Outside?

Problem

In the diagram above, $\triangle PQR$ is an equilateral triangle with sides of length 2 cm. Arc PQ is an arc of a circle with centre R and radius RQ . Arc PR and arc RQ are similarly drawn with centres Q and P , respectively. Determine the total area of the shaded regions in the diagram.

Solution

First, determine the area of $\triangle PQR$. Since the triangle is equilateral, each of the angles in the triangle are 60° . Construct altitude PT , shown to the right. Then $\triangle PTR$ is a 30° - 60° - 90° triangle with sides in the ratio $1 : \sqrt{3} : 2$.



Since $QR = 2$ cm, it follows that $TR = 1$ cm and $PT = \sqrt{3}$ cm. Therefore,

$$\begin{aligned} \text{area } \triangle PQR &= (QR)(PT) \div 2 \\ &= (2)(\sqrt{3}) \div 2 \\ &= \sqrt{3} \text{ cm}^2 \end{aligned}$$

The diagram consists of three overlapping circle sectors, one with centre P , one with centre Q , and one with centre R . Each circle sector has the same radius, 2 cm, and a 60° central angle. Therefore each sector has the same area, $60 \div 360$ or one-sixth the area of a circle of radius 2 cm. That is,

$$\text{area of each sector} = \frac{1}{6}\pi r^2 = \frac{1}{6}\pi(2)^2 = \frac{2}{3}\pi \text{ cm}^2$$

The shaded part of each circle sector is equal to the area of the sector minus the area of $\triangle PQR$. Since there are three congruent shaded areas,

$$\begin{aligned} \text{total shaded area} &= 3(\text{area of any whole circle sector} - \text{area of } \triangle PQR) \\ &= 3\left(\frac{2}{3}\pi - \sqrt{3}\right) \\ &= (2\pi - 3\sqrt{3}) \text{ cm}^2 \end{aligned}$$

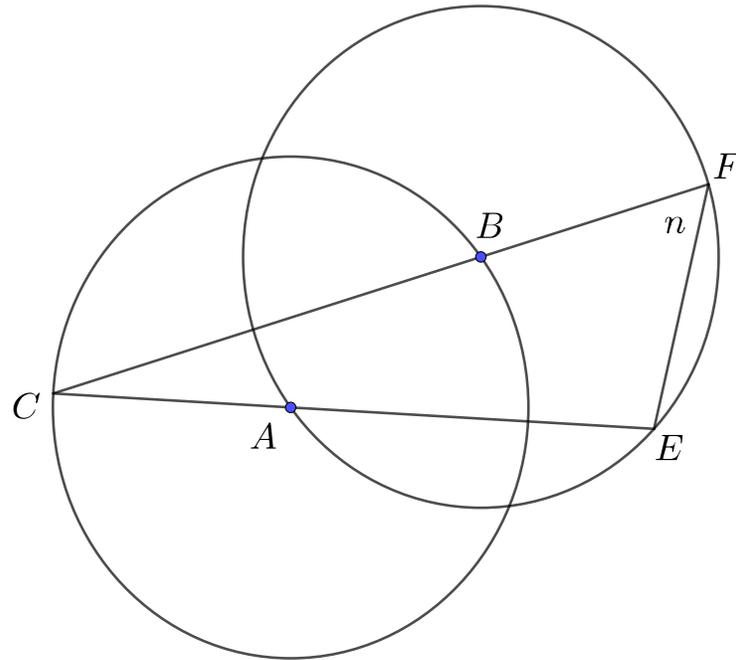
Therefore, the total area of the shaded regions is equal to $(2\pi - 3\sqrt{3}) \text{ cm}^2$.



Problem of the Week

Problem E

The Angle Chase is On



Two intersecting circles are constructed so that the centre of one circle is on the circumference of the other circle. That is, A is the centre of a circle which passes through B and B is the centre of a circle which passes through A . CAE and CBF are straight line segments.

If $\angle F = n$, with $0^\circ < n < 90^\circ$, determine the measure of $\angle C$ in terms of n .





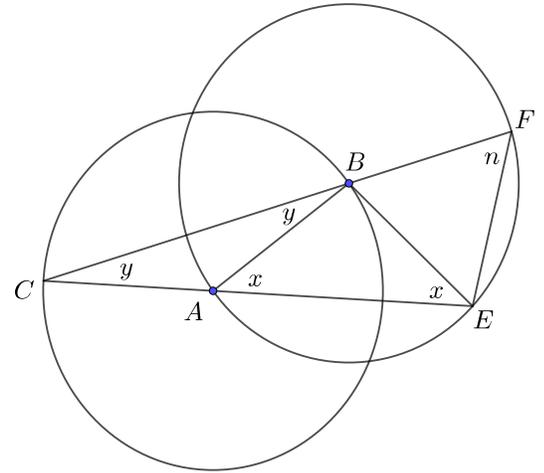
Problem of the Week

Problem E and Solution

The Angle Chase is On

Problem

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Solution

Construct AB and BE . A , E and F are on the circumference of the circle with centre B . Therefore, $BA = BE = BF$.

In $\triangle BEF$, $BE = BF$. Then $\triangle BEF$ is isosceles and $\angle BEF = \angle F = n$.

Let $\angle BEA = x$ and $\angle ABC = y$.

In $\triangle BAE$, $BA = BE$ and the triangle is isosceles. Therefore, $\angle BAE = \angle BEA = x$.

Since B and C are on the circle with centre A , $AC = AB$ and $\triangle ABC$ is isosceles. Therefore, $\angle C = \angle ABC = y$.

$\angle EAB$ is an exterior angle to $\triangle ABC$. By the exterior angle theorem for triangles, $\angle EAB = \angle C + \angle ABC$. But $\angle EAB = x$ and $\angle C + \angle ABC = y + y = 2y$. Therefore, $x = 2y$.

In $\triangle CEF$,

$$\angle C + \angle E + \angle F = 180^\circ$$

$$\angle C + \angle CEB + \angle BEF + n = 180^\circ$$

$$y + x + n + n = 180^\circ$$

Also, $x = 2y$. Therefore,

$$y + 2y + 2n = 180^\circ$$

$$3y = 180^\circ - 2n$$

$$y = \frac{180^\circ - 2n}{3}$$

$$y = 60^\circ - \frac{2}{3}n$$

Therefore, $\angle C = 60^\circ - \frac{2}{3}n$.



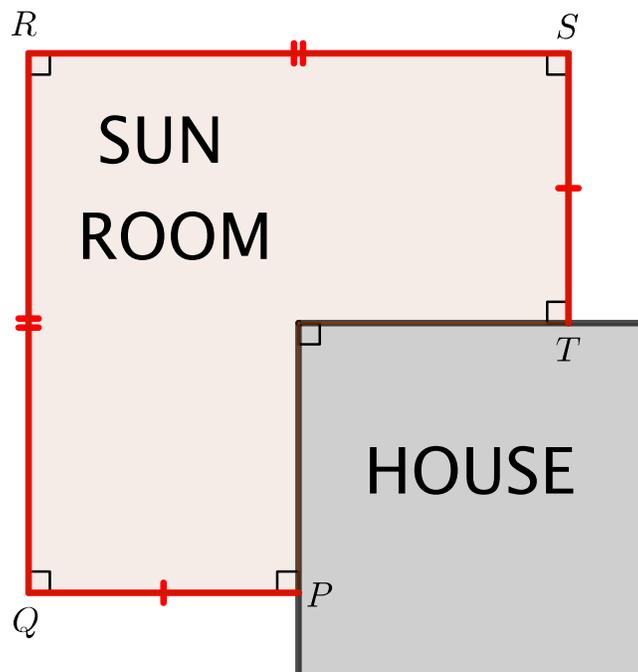
Problem of the Week

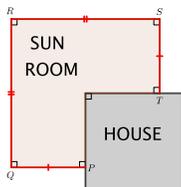
Problem E

Adding On

A sun room is to be built out from one of the corners of a house, as illustrated below. The lengths of the two longer outer walls, RS and QR , are to be the same, and the lengths of the two shorter outer walls, ST and PQ , are to be the same. These walls will be at right angles to each other and to the existing house. The total distance from P to Q to R to S to T is to be 30 m. Determine the dimensions that will give the maximum sun room area.

The total distance from P to Q to R to S to T is to be 30 m. Determine the dimensions that will give the maximum sun room area.





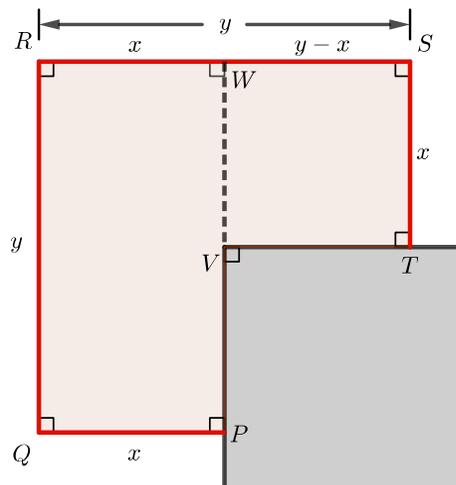
Problem of the Week

Problem E and Solution

Adding On

Problem

A sun room is to be built out from one of the corners of a house, as illustrated above. The lengths of the two longer outer walls, RS and QR , are to be the same, and the lengths of the two shorter outer walls, ST and PQ , are to be the same. These walls will be at right angles to each other and to the existing house. The total distance from P to Q to R to S to T is to be 30 m. Determine the dimensions that will give the maximum sun room area.



Solution

Label the corner of the house V . Draw a line segment through PV to RS intersecting at W . Then $PW \perp RS$ and this makes two rectangles, $PQRW$ and $WSTV$.

Let x represent the lengths of both PQ and ST . Let y represent the lengths of both QR and RS . Since $PQRW$ is a rectangle, $RW = PQ = x$ and $WS = RS - RW = y - x$.

Let A represent the area of the sun room.

The total distance from P to Q to R to S to T is

$$PQ + QR + RS + ST = x + y + y + x = 2x + 2y = 30$$

Dividing by 2, we get $x + y = 15$. Rearranging to solve for y , we obtain $y = 15 - x$. (1)

$$\begin{aligned} \text{Area of sun room} &= \text{Area } PQRW + \text{Area } WSTV \\ &= QR \times RW + WS \times ST \\ &= yx + (y - x)x \\ &= (15 - x)x + ((15 - x) - x)x, \quad \text{substituting from (1) above} \\ &= (15 - x)x + (15 - 2x)x \\ &= 15x - x^2 + 15x - 2x^2 \\ &= -3x^2 + 30x \\ &= -3(x^2 - 10x) \\ &= -3(x^2 - 10x + 5^2 - 5^2), \quad \text{by completing the square} \\ &= -3(x^2 - 10x + 25) + 75 \end{aligned}$$

$$\therefore A = -3(x - 5)^2 + 75$$

This is the equation of a parabola which opens down from a vertex of $(5, 75)$. The maximum area is 75 m^2 when $x = 5$ m. When $x = 5$, $y = 15 - x = 15 - 5 = 10$ m.

Therefore, if the longer side of the sun room is 10 m and the shorter side of the sun room is 5 m, this gives a maximum area of 75 m^2 .



Problem of the Week

Problem E

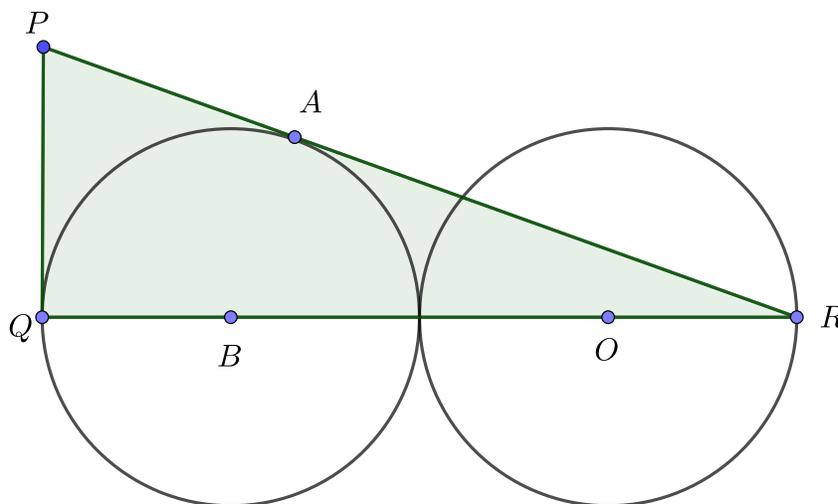
Two Circles and a Triangle

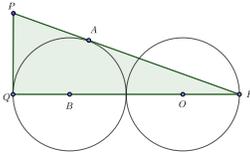
Two circles, with centres O and B and each with a radius of 2, are tangent to each other. A straight line is drawn through O and B meeting the circles at Q and R . Two other sides of $\triangle PQR$ are drawn such that side PR is tangent to the circle with centre B at A and side PQ is tangent to the circle with centre O at Q .

Determine the length of PQ .

It may be helpful to use the facts that

- a line drawn from the centre of a circle to a point of tangency is perpendicular to the tangent, and
- if two circles are tangent to each other, a line segment joining the two centres passes through the point of tangency.





Problem of the Week

Problem E and Solution

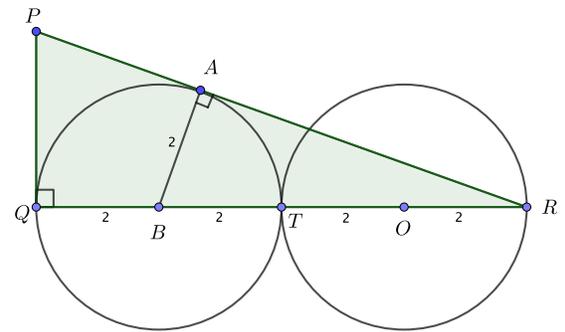
Two Circles and a Triangle

Problem

Two circles, with centres O and B and each with a radius of 2, are tangent to each other. A straight line is drawn through O and B meeting the circles at Q and R . Two other sides of $\triangle PQR$ are drawn such that side PR is tangent to the circle with centre B at A and side PQ is tangent to the circle with centre O at Q . Determine the length of PQ .

Solution

Let T be the point of tangency of the two circles. Then T lies on OB , the line segment joining the two centres. Also, $QB = BT = TO = OR = 2$. Since PR is tangent to the circle with centre B at A , $AB \perp PR$. Since PQ is tangent to the circle with centre O at Q , $PQ \perp QB$. Since AB and QB are radii of the circle with centre B , $QB = AB = 2$. This information has been added to the diagram.



$\triangle BAR$ is right angled at A since $AB \perp AR$. Therefore,
 $AR^2 = BR^2 - AB^2 = 6^2 - 2^2 = 36 - 4 = 32$. Since $AR > 0$, $AR = \sqrt{32} = 4\sqrt{2}$.

At this point there are many ways to find the length of PQ . We will look at three solutions.

Solution 1

An interesting (and possibly different) way to find PQ is to use basic trigonometry in the right triangles. In $\triangle BAR$, $\tan(R) = \frac{AB}{AR}$ and in $\triangle PQR$, $\tan(R) = \frac{PQ}{QR}$.

$$\begin{aligned} \therefore \frac{AB}{AR} &= \frac{PQ}{QR} \\ \frac{2}{4\sqrt{2}} &= \frac{PQ}{8} \\ PQ &= \frac{16}{4\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2} \end{aligned}$$

Therefore, the length of PQ is $2\sqrt{2}$.



Solution 2

Construct a line from B to P . We will show that $PQ = PA$.

Both $\triangle PQB$ and $\triangle PAB$ are right triangles. Using the Pythagorean Theorem,

$$PB^2 = PQ^2 + QB^2 \quad \text{and} \quad PB^2 = PA^2 + AB^2$$

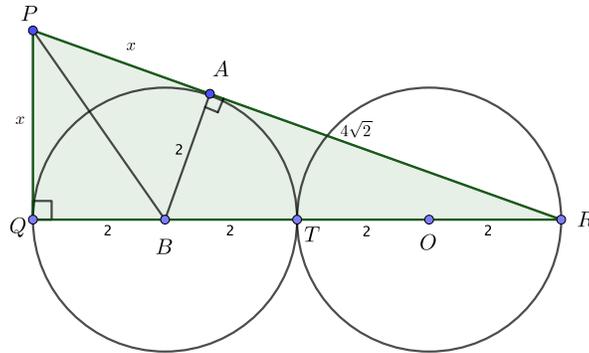
$$\therefore PQ^2 + QB^2 = PA^2 + AB^2$$

$$PQ^2 + (2)^2 = PA^2 + (2)^2$$

It follows that $PQ^2 = PA^2$

And $PQ = PA$, since $PQ > 0$ and $PA > 0$

Let $PQ = PA = x$. The information is added to the diagram.



Since $PQ \perp QR$, $\triangle PQR$ is a right triangle with $PQ = x$, $QR = QB + BT + TO + OR = 8$ and $PR = PA + AR = 4\sqrt{2} + x$.

Using the Pythagorean Theorem in $\triangle PQR$,

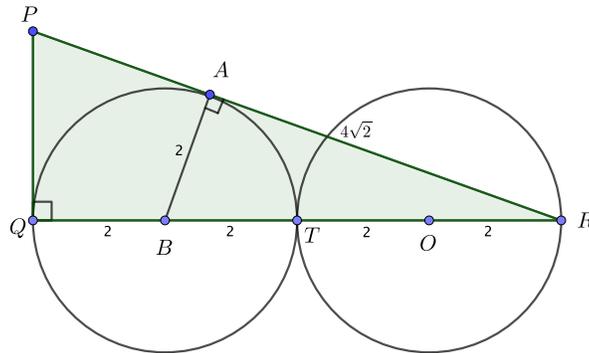
$$\begin{aligned} PR^2 &= QR^2 + PQ^2 \\ (4\sqrt{2} + x)^2 &= 8^2 + x^2 \\ 32 + (8\sqrt{2})x + x^2 &= 64 + x^2 \\ (8\sqrt{2})x &= 32 \\ x &= \frac{32}{8\sqrt{2}} \\ x &= \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ x &= 2\sqrt{2} \end{aligned}$$

Therefore, the length of PQ is $2\sqrt{2}$.



Solution 3

We will first show that $\triangle BAR$ is similar to $\triangle PQR$.



$$\begin{aligned}
 \text{In } \triangle BAR \quad \text{and} \quad \triangle PQR \\
 \angle ARB &= \angle QRP \quad (\text{common angle}) \\
 \angle BAR &= \angle PQR \quad (\text{both are right angles}) \\
 \therefore \triangle BAR &\sim \triangle PQR \quad (\text{AA } \triangle \text{ similarity}) \\
 \text{and } \frac{AR}{QR} &= \frac{AB}{PQ} \text{ follows.} \\
 \frac{4\sqrt{2}}{8} &= \frac{2}{PQ} \\
 (4\sqrt{2})PQ &= 16 \\
 PQ &= \frac{16}{4\sqrt{2}} \\
 &= \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
 &= 2\sqrt{2}
 \end{aligned}$$

Therefore, the length of PQ is $2\sqrt{2}$.



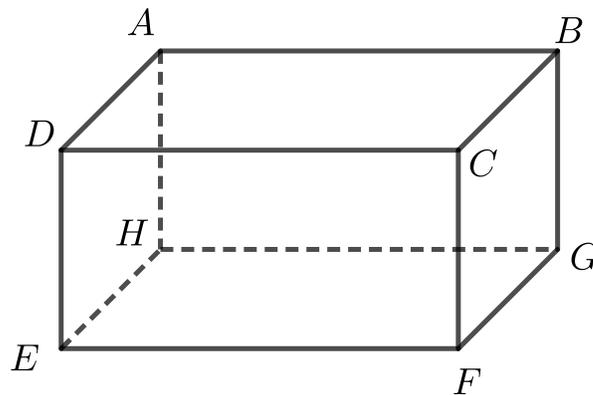
Problem of the Week

Problem E

Body Diagonal

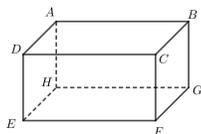
In the given rectangular prism $ABCDEFGH$,

- the sum of the lengths of all of the edges is 28 cm, and
- the total surface area is 13 cm^2 .



What is the length of the diagonal EB ?





Problem of the Week

Problem E and Solution

Body Diagonal

Problem

In the above rectangular prism $ABCDEFGH$,

- the sum of the lengths of all of the edges is 28 cm, and
- the total surface area is 13 cm^2 .

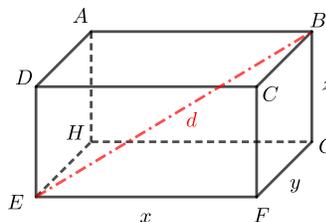
What is the length of the diagonal EB ?

Solution

Let $EF = x$, $FG = y$ and $BG = z$.

We construct EB and label it d .

The updated diagram is shown to the right.



By the Pythagorean Theorem in $\triangle EBG$, $EB^2 = EG^2 + BG^2$.

By the Pythagorean Theorem in $\triangle EFG$, $EG^2 = EF^2 + FG^2$.

Therefore, $EB^2 = EG^2 + BG^2 = EF^2 + FG^2 + BG^2$.

That is, $d^2 = x^2 + y^2 + z^2$.

Since the sum of the lengths of all the edges is 28, then $4x + 4y + 4z = 28$ or $x + y + z = 7$.

Since the surface area of the prism is 13, we know $2xy + 2yz + 2xz = 13$.

Since we have squared terms and pair factor terms it might be helpful to expand $(x + y + z)^2$.

$$\begin{aligned} (x + y + z)^2 &= (x + (y + z))^2 \\ &= x^2 + 2x(y + z) + (y + z)^2 \\ &= x^2 + 2xy + 2xz + y^2 + 2xy + z^2 \end{aligned}$$

Therefore,

$$\underbrace{(x + y + z)^2}_{7^2} = \underbrace{x^2 + y^2 + z^2}_{d^2} + \underbrace{2xy + 2xz + 2yz}_{13}$$

Substituting what we know,

$$49 - 13 = d^2$$

$$d^2 = 36$$

$$d = 6, \text{ since } d > 0$$

Therefore, the length of EB is 6 cm.



Problem of the Week

Problem E

Uphill then Down

Two racers compete in a 1400 m race. The first 700 m of the race is uphill, and the second 700 m is down the same hill along the same path. Each racer's constant uphill speed is half of their respective constant downhill speed.

The faster racer reaches the top of the hill and immediately starts running downhill. The faster racer then meets the slower racer (still going uphill) 70 m from the top of the hill.

When the faster racer finishes the race, how far is the slower racer from the finish line?





Problem of the Week

Problem E and Solution

Uphill then Down

Problem

Two racers compete in a 1400 m race. The first 700 m of the race is uphill, and the second 700 m is down the same hill along the same path. Each racer's constant uphill speed is half of their respective constant downhill speed.

The faster racer reaches the top of the hill and immediately starts running downhill. The faster racer then meets the slower racer (still going uphill) 70 m from the top of the hill.

When the faster racer finishes the race, how far is the slower racer from the finish line?

Solution

Let a represent the faster racer's uphill speed and $2a$ represent their downhill speed.

Let b represent the slower racer's uphill speed and $2b$ represent their downhill speed.

We know that *distance* = *speed* \times *time* or *time* = *distance* \div *speed*.

When the faster racer and the slower racer meet 70 m from the top of the hill, their times are the same. At this time the slower racer has run $700 - 70 = 630$ m uphill. Therefore:

$$\text{faster's time up 700 m} + \text{faster's time down 70 m} = \text{slower's time up 630 m}$$

$$\frac{700}{a} + \frac{70}{2a} = \frac{630}{b}$$

Reducing the second fraction: $\frac{700}{a} + \frac{35}{a} = \frac{630}{b}$

Simplifying: $\frac{735}{a} = \frac{630}{b}$

By rearranging, we obtain $\frac{a}{b} = \frac{735}{630} = \frac{7}{6}$. (1)

Solution continues on the next page.



Let x represent the distance the slower racer is behind the faster racer at the end of the race.

When the faster racer reaches the finish, they have travelled 700 m uphill at a speed of a and 700 m down at a speed of $2a$. The slower racer has travelled 700 m uphill at a speed of b and $(700 - x)$ m down at a speed of $2b$.

When the faster racer reaches the finish line and the slower racer is x m from the finish, their times are the same. Therefore,

$$\begin{aligned}\frac{700}{a} + \frac{700}{2a} &= \frac{700}{b} + \frac{700 - x}{2b} \\ \frac{1400}{2a} + \frac{700}{2a} &= \frac{1400}{2b} + \frac{700 - x}{2b} \\ \frac{2100}{2a} &= \frac{2100 - x}{2b}\end{aligned}$$

Multiplying both sides by 2 ,

$$\frac{2100}{a} = \frac{2100 - x}{b}$$

It follows that $\frac{a}{b} = \frac{2100}{2100 - x}$. But from (1) earlier, $\frac{a}{b} = \frac{7}{6}$.

$$\text{Therefore, } \frac{2100}{2100 - x} = \frac{7}{6}$$

$$\begin{aligned}\text{Solving for } x, \quad 12\,600 &= 14\,700 - 7x \\ 7x &= 2100 \\ x &= 300\end{aligned}$$

Therefore, the slower racer is 300 m from the finish line when the faster racer finishes the race.



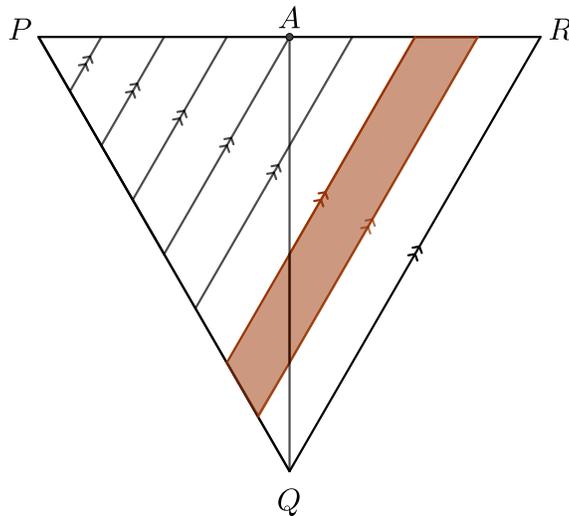
Problem of the Week

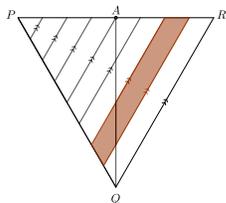
Problem E

Shady Division

$\triangle PQR$ is an equilateral triangle with sides of length 16 cm. Two sides of the triangle, PR and PQ , are each divided into 8 segments of equal length. Each point of division on PR is connected to its corresponding point of division on PQ , creating 7 line segments. The region formed between two of these line segments is shaded, as shown.

An altitude is constructed from Q to A on PR , dividing the shaded region into two parts. In this shaded region, determine the ratio of the area on the left side of the altitude to the area on the right side of the altitude.





Problem of the Week

Problem E and Solution

Shady Division

Problem

$\triangle PQR$ is an equilateral triangle with sides of length 16 cm. Two sides of the triangle, PR and PQ , are each divided into 8 segments of equal length. Each point of division on PR is connected to its corresponding point of division on PQ , creating 7 line segments. The region formed between two of these line segments is shaded, as shown. An altitude is constructed from Q to A on PR , dividing the shaded region into two parts. In this shaded region, determine the ratio of the area on the left side of the altitude to the area on the right side of the altitude.

Solution

Solution 1:

Since PR and PQ are each divided into 8 equal segments, then each segment will have a length of $16 \div 8 = 2$ cm.

Label the shaded region $DEFG$. Let B and C be where altitude QA intersects $DEFG$, as shown to the right. We are required to find the ratio of the area of $DBCG$ to the area of $CBEF$. We will find the area of $DEFG$ by finding the area of $\triangle DPE$ and subtracting it from the area of $\triangle GPF$.

We first mention a few facts that we will use in our solution. For some equilateral $\triangle WXY$ with side length 2, altitude WZ right bisects base XY and it follows that $XZ = ZY = 1$. Using the Pythagorean Theorem, the length of WZ is $\sqrt{2^2 - 1^2} = \sqrt{3}$. In any equilateral triangle, the ratio of the height to the side length is $\sqrt{3} : 2$. In other words, the height of any equilateral triangle is $\frac{\sqrt{3}}{2}$ times its side length. The altitude WZ also bisects $\angle XWY$, so $\angle XWZ = \angle YWZ = 30^\circ$ and $\triangle WXZ$ is a 30° - 60° - 90° triangle whose sides are in the ratio $1 : \sqrt{3} : 2$.

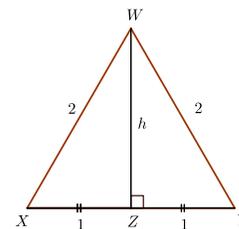
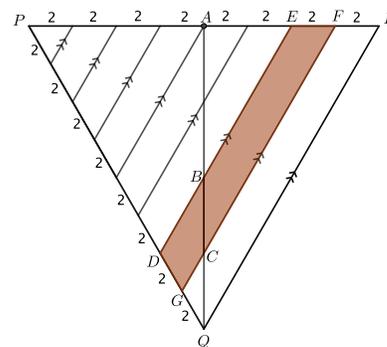
Now back to given $\triangle PQR$, we know that $\angle QPR = \angle PRQ = \angle RQP = 60^\circ$.

In $\triangle DPE$, $\angle DPE = \angle QPR = 60^\circ$ since they are the same angle. Since $DE \parallel RQ$, $\angle PED = \angle PRQ = 60^\circ$ and $\angle EDP = \angle RQP = 60^\circ$. Since all of the angles in $\triangle DPE$ are 60° , it is an equilateral triangle with sides of length 12 cm. Using our above result, the height of $\triangle DPE = \frac{\sqrt{3}}{2} \times 12 = 6\sqrt{3}$. The area of $\triangle DPE = \frac{12 \times 6\sqrt{3}}{2} = 36\sqrt{3}$ cm².

In a similar way, we can show that $\triangle GPF$ is equilateral with side length 14 cm. The height of $\triangle GPF = \frac{\sqrt{3}}{2} \times 14 = 7\sqrt{3}$ and the area of $\triangle GPF = \frac{14 \times 7\sqrt{3}}{2} = 49\sqrt{3}$ cm².

Therefore,

$$\begin{aligned} \text{area of } DEFG &= \text{area of } \triangle GPF - \text{area of } \triangle DPE \\ &= 49\sqrt{3} - 36\sqrt{3} \\ &= 13\sqrt{3} \text{ cm}^2 \end{aligned}$$





Next, we find the area of $CBEF$ by finding the area of $\triangle BEA$ and subtracting it from the area of $\triangle CFA$.

In $\triangle BEA$, $\angle BAE = \angle QAR = 90^\circ$, since the altitude QA is perpendicular to base PR and the two angles represent the same angle. Since $DE \parallel QR$, $\angle BEA = \angle QRP = 60^\circ$. It follows that $\angle EBA = 30^\circ$. Therefore, $\triangle BEA$ is a $30^\circ - 60^\circ - 90^\circ$ triangle and $EA : BA : BE = 1 : \sqrt{3} : 2$. Since $EA = 4$, it follows that $BA = 4\sqrt{3}$ and $BE = 8$. The area of $\triangle BEA = \frac{4 \times 4\sqrt{3}}{2} = 8\sqrt{3} \text{ cm}^2$.

In a similar way, we can show that $\triangle CFA$ is a $30^\circ - 60^\circ - 90^\circ$ triangle with $FA : CA : CF = 1 : \sqrt{3} : 2$. Since $FA = 6$, it follows that $CA = 6\sqrt{3}$ and $CF = 12$. The area of $\triangle CFA = \frac{6 \times 6\sqrt{3}}{2} = 18\sqrt{3} \text{ cm}^2$.

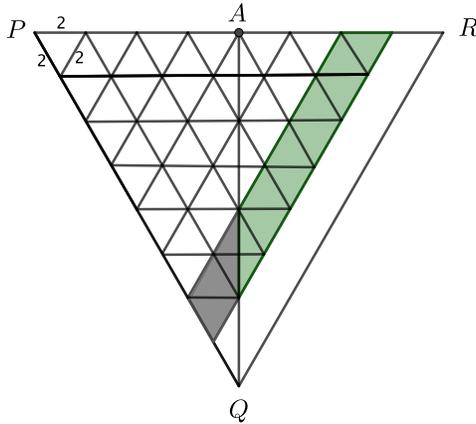
Now, $\text{area of } CBEF = \text{area of } \triangle CFA - \text{area of } \triangle BEA = 18\sqrt{3} - 8\sqrt{3} = 10\sqrt{3} \text{ cm}^2$
 $\text{area of } DBCG = \text{area of } DEFG - \text{area of } CBEF = 13\sqrt{3} - 10\sqrt{3} = 3\sqrt{3} \text{ cm}^2$

Therefore, the ratio of the area of $DBCG$ to the area of $CBEF$ is $3\sqrt{3} : 10\sqrt{3}$ or $3 : 10$.

Solution 2:

In the diagram, we see that we can tile $\triangle QPR$ with the top left equilateral triangle of side length 2 cm. (You may wish to justify that this is indeed an equilateral triangle.)

Then, three equilateral triangles fit in the first trapezoid from the left, 5 equilateral triangles fit in the second trapezoid from the left, 7 equilateral triangles fit in the third trapezoid from the left, 9 equilateral triangles fit in the fourth trapezoid from the left, 11 equilateral triangles fit in the fifth trapezoid from the left, and 13 equilateral triangles fit in the sixth trapezoid from the left.



The shaded trapezoid is the sixth trapezoid from the left and so contains a total of 13 congruent equilateral triangles. To the left of the altitude there are $2 + \frac{1}{2} + \frac{1}{2} = 3$ of the triangles and to the right of the altitude there are $13 - 3 = 10$ of the triangles. Each of the 13 triangles in the shaded trapezoid have the same area.

The ratio of the shaded area to the left of the altitude to the shaded area to the right of the altitude is $3 : 10$.

There are many things in this second solution that the solver may wish to justify.

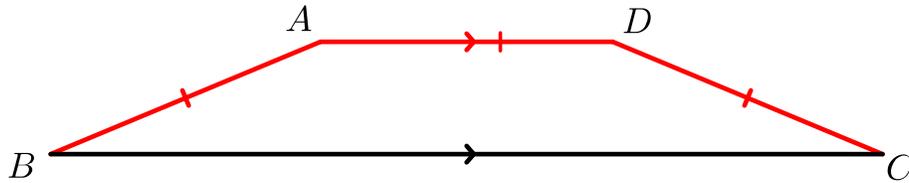


Problem of the Week

Problem E

Three Equal Sides

In trapezoid $ABCD$, the lengths of AB , AD and DC are equal and the length of BC is 2 units less than the sum of the lengths of the other three sides.



If the distance between the parallel sides AD and BC is 5 units, what is the area of the trapezoid?

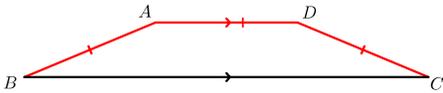




Problem of the Week

Problem E and Solution

Three Equal Sides

**Problem**

In trapezoid $ABCD$, the lengths of AB , AD and DC are equal and the length of BC is 2 units less than the sum of the lengths of the other three sides. If the distance between the parallel sides AD and BC is 5 units, what is the area of the trapezoid?

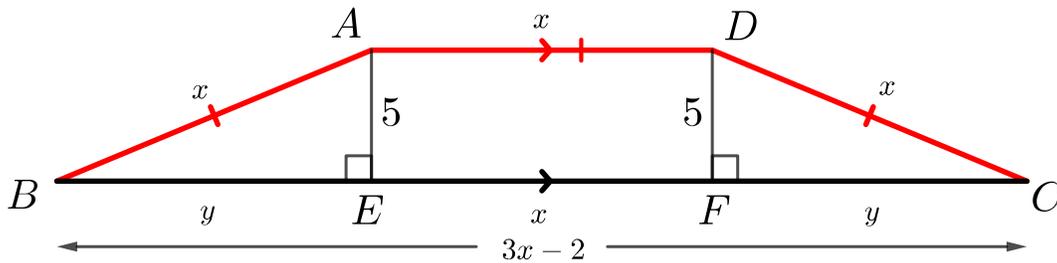
Solution

Let x represent the length of AB . Then $AB = AD = DC = x$. Since the base BC is two less than the sum of the three equal sides, $BC = 3x - 2$.

Construct altitudes from A and D meeting BC at E and F , respectively. Then $AE = DF = 5$, the distance between the two parallel sides.

Let y represent the length of BE . We can show that $BE = FC$ using the Pythagorean Theorem as follows: $BE^2 = AB^2 - AE^2 = x^2 - 5^2 = x^2 - 25$ and $FC^2 = DC^2 - DF^2 = x^2 - 5^2 = x^2 - 25$. Then $FC^2 = x^2 - 25 = BE^2$, so $FC = BE = y$ since $FC > 0$.

Since $\angle AEF = \angle DFE = 90^\circ$ and AD is parallel to EF , it follows that $\angle DAE = \angle ADF = 90^\circ$ and $AEFD$ is a rectangle so $EF = AD = x$. The following diagram contains all of the given and found information.



We can now determine a relationship between x and y .

$$\begin{aligned} BC &= BE + EF + FC \\ 3x - 2 &= y + x + y \\ 2x - 2 &= 2y \\ x - 1 &= y \end{aligned}$$

In right $\triangle ABE$, $AB^2 = BE^2 + AE^2$ gives $x^2 = y^2 + 5^2$. Substituting $y = x - 1$ we get $x^2 = (x - 1)^2 + 25$. Solving, $x^2 = x^2 - 2x + 1 + 25$, or $2x = 26$, or $x = 13$.

Since $x = 13$, $3x - 2 = 3(13) - 2 = 37$. Therefore, $AD = x = 13$ and $BC = 3x - 2 = 37$.

Therefore,

$$\begin{aligned} \text{area of trapezoid } ABCD &= AE \times (AD + BC) \div 2 \\ &= 5 \times (13 + 37) \div 2 \\ &= 125 \text{ units}^2 \end{aligned}$$



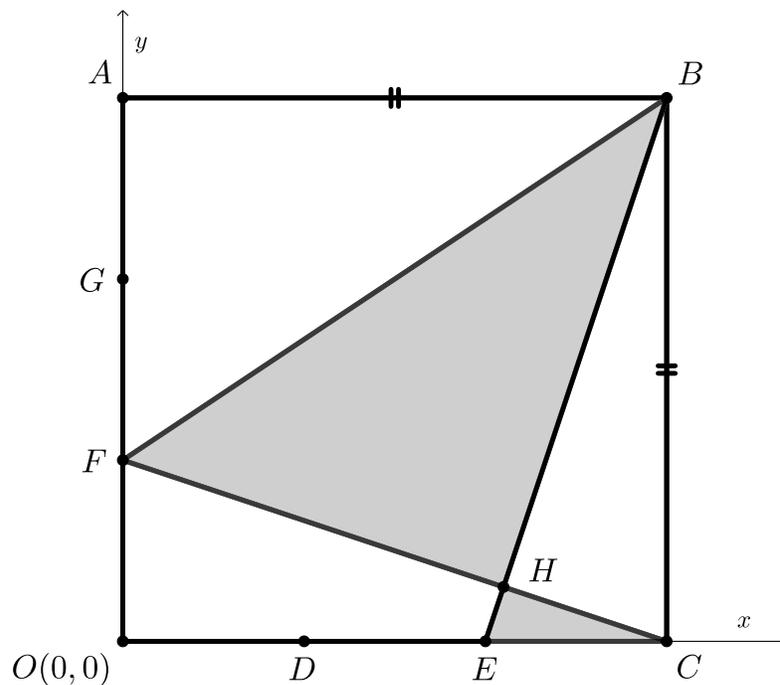
Problem of the Week

Problem E

Maybe One-Third?

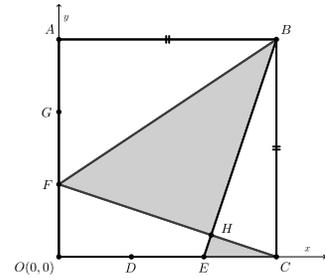
In the diagram, square $OABC$ is positioned with O at the origin $(0,0)$, A on the positive y -axis, C on the positive x -axis, and B in the first quadrant. Side OA is trisected by points F and G so that $OF = FG = GA = 100$. Side OC is trisected by points D and E so that $OD = DE = EC = 100$. Line segment BE intersects line segment CF at H .

If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?





Problem of the Week Problem E and Solution Maybe One-Third?



Problem

In the diagram, square $OABC$ is positioned with O at the origin $(0, 0)$, A on the positive y -axis, C on the positive x -axis, and B in the first quadrant. Side OA is trisected by points F and G so that $OF = FG = GA = 100$. Side OC is trisected by points D and E so that $OD = DE = EC = 100$. Line segment BE intersects line segment CF at H . If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?

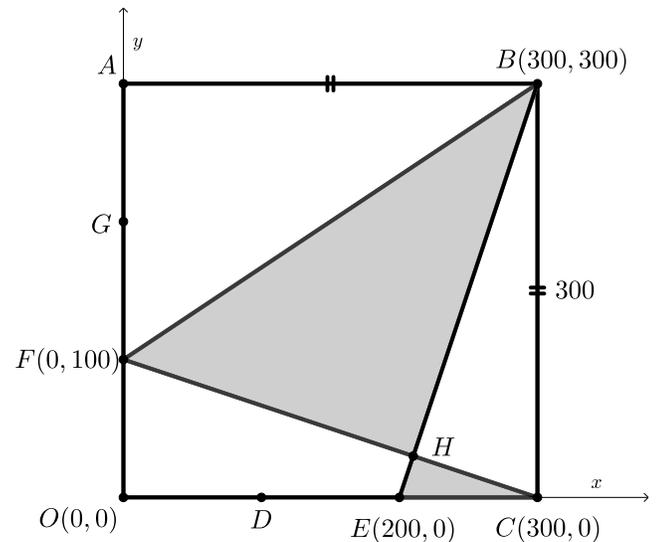
Solution

Since $OF = 100$ and F is on the positive y -axis, the coordinates of F are $(0, 100)$.

Since $OD = DE = 100$, it follows that $OE = 200$. Since E is on the positive x -axis, the coordinates of E are $(200, 0)$.

Since $OD = DE = EC = 100$, it follows that the side length of the square is $OC = 300$. Since C is on the positive x -axis, the coordinates of C are $(300, 0)$.

It then follows that the coordinates of B are $(300, 300)$.



The diagram has been updated to reflect the new information.

We will proceed to find the coordinates of H .

Find the equation of the line through $B(300, 300)$ and $E(200, 0)$.

The slope of $BE = \frac{300-0}{300-200} = 3$. We substitute $x = 200$, $y = 0$ and $m = 3$ into $y = mx + b$ to find b . Then $0 = 3(200) + b$ and $b = -600$ follows. The equation of the line through BE is $y = 3x - 600$. (1)

Find the equation of the line through $C(300, 0)$ and $F(0, 100)$.

The slope of $CF = \frac{100-0}{0-300} = -\frac{1}{3}$. Since $F(0, 100)$ is on the y -axis, the y -intercept is 100. It follows that the equation of the line through C and F is $y = -\frac{1}{3}x + 100$. (2)



Find the coordinates of H , the intersection of the two lines.

At the intersection, the x -coordinates are equal and the y -coordinates are equal. In (1) and (2), since $y = y$, then

$$3x - 600 = -\frac{1}{3}x + 100 \Rightarrow 9x - 1800 = -x + 300 \Rightarrow 10x = 2100 \Rightarrow x = 210$$

Substituting $x = 210$ into (1), $y = 3(210) - 600 = 30$. Therefore, the coordinates of H are $(210, 30)$.

At this point we could follow one of two approaches. The first approach would be to find the area of the shaded regions indirectly, by first determining the area of the unshaded regions and then subtracting this from the area of the square. We will leave this approach to the solver.

Our second approach, which is below, is to calculate the areas of the shaded triangles directly.

The slope of BE is 3 and the slope of CF is $-\frac{1}{3}$. Since these slopes are negative reciprocals, we know that $BE \perp CF$. It follows that $\triangle BHF$ and $\triangle CHE$ are right-angled triangles.

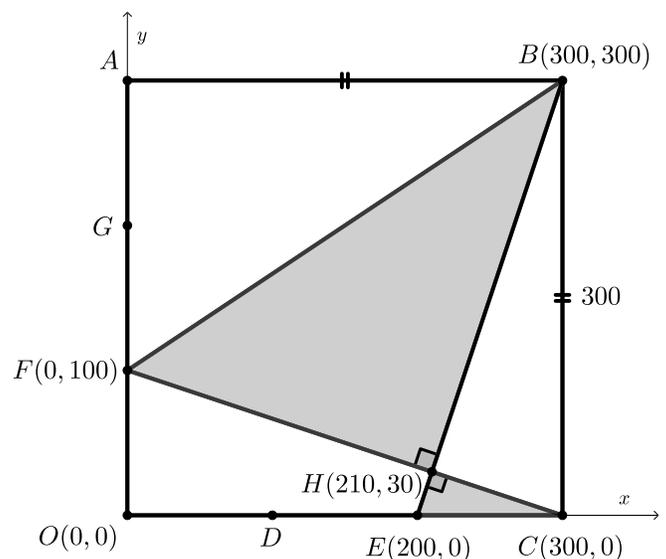
We will now proceed with finding the side lengths necessary to calculate the area of each shaded triangle.

We first find the area of $\triangle BHF$.

$$\begin{aligned} BH &= \sqrt{(300 - 210)^2 + (300 - 30)^2} \\ &= \sqrt{90^2 + 270^2} \\ &= \sqrt{90^2(1 + 3^2)} \\ &= 90\sqrt{10} \end{aligned}$$

$$\begin{aligned} HF &= \sqrt{(0 - 210)^2 + (100 - 30)^2} \\ &= \sqrt{210^2 + 70^2} \\ &= \sqrt{70^2(3^2 + 1)} \\ &= 70\sqrt{10} \end{aligned}$$

$$\begin{aligned} \text{Area } \triangle BHF &= BH \times HF \div 2 \\ &= 90\sqrt{10} \times 70\sqrt{10} \div 2 \\ &= 31\,500 \end{aligned}$$



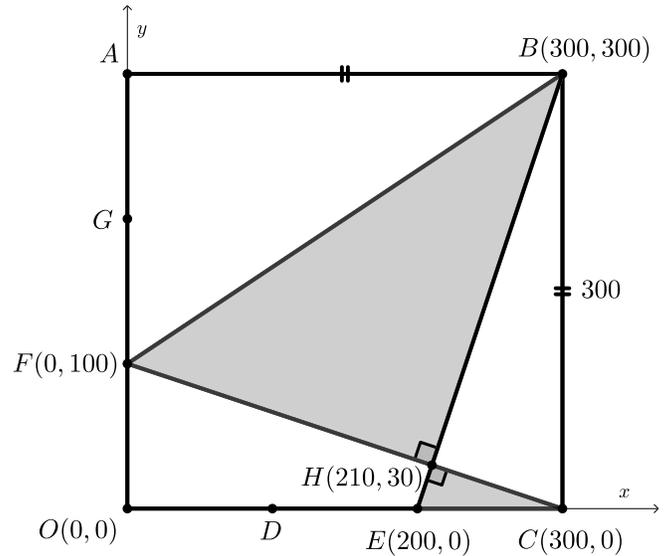


Next we find the area of $\triangle CHE$.

$$\begin{aligned} CH &= \sqrt{(300 - 210)^2 + (0 - 30)^2} \\ &= \sqrt{90^2 + 30^2} \\ &= \sqrt{30^2(3^2 + 1)} \\ &= 30\sqrt{10} \end{aligned}$$

$$\begin{aligned} HE &= \sqrt{(210 - 200)^2 + (30 - 0)^2} \\ &= \sqrt{10^2 + 30^2} \\ &= \sqrt{10^2(1 + 3^2)} \\ &= 10\sqrt{10} \end{aligned}$$

$$\begin{aligned} \text{Area } \triangle CHE &= CH \times HE \div 2 \\ &= 30\sqrt{10} \times 10\sqrt{10} \div 2 \\ &= 1500 \end{aligned}$$



We can now calculate the total area shaded, the area of square $OABC$, and the fraction of the area of the square that is shaded.

$$\begin{aligned} \text{Total Area Shaded} &= \text{Area } \triangle BHF + \text{Area } \triangle CHE \\ &= 31\,500 + 1500 \\ &= 33\,000 \end{aligned}$$

$$\begin{aligned} \text{Area } OABC &= OA \times OC \\ &= 300 \times 300 \\ &= 90\,000 \end{aligned}$$

$$\begin{aligned} \text{Fraction of Total Area Shaded} &= \frac{\text{Area } \triangle BHF}{\text{Area } OABC} \\ &= \frac{33\,000}{90\,000} \\ &= \frac{11}{30} \end{aligned}$$

Therefore, $\frac{11}{30}$ of the total area of the square is shaded. This, in fact, is more than one-third.

Algebra (A)





Problem of the Week

Problem E

Average Again

Erin has written three tests for her math class. She calculates the average of her marks on the first two tests. This average is then averaged with her third test mark to get 80%. She then calculates the average of her marks on the last two tests. This average is then averaged with her first test mark to get 83.5%. Finally, she calculates the average of her marks on the first and third tests. This average is then averaged with her second test mark to get 84.5%.

Her fourth test is coming up and after writing this test she wants her overall average for the four tests to be exactly 86%. If all four tests are out of 100 marks, what mark does Erin need to get on her fourth test in order for her overall average to be exactly 86%?





Problem of the Week

Problem E and Solution

Average Again

Problem

Erin has written three tests for her math class. She calculates the average of her marks on the first two tests. This average is then averaged with her third test mark to get 80%. She then calculates the average of her marks on the last two tests. This average is then averaged with her first test mark to get 83.5%. Finally, she calculates the average of her marks on the first and third tests. This average is then averaged with her second test mark to get 84.5%. Her fourth test is coming up and after writing this test she wants her overall average for the four tests to be exactly 86%. If all four tests are out of 100 marks, what mark does Erin need to get on her fourth test in order for her overall average to be exactly 86%?

Solution

Let a represent Erin's first test mark, b represent Erin's second test mark, c represent Erin's third test mark and d represent Erin's fourth test mark.

When the average of her first and second marks is averaged with her third test mark, the new average is 80 so $\frac{\frac{a+b}{2}+c}{2} = 80$. Multiplying by 2 gives $\frac{a+b}{2} + c = 160$. Multiplying by 2 again yields $a + b + 2c = 320$. (1)

When the average of her second and third marks is averaged with her first test mark, the new average is 83.5 so $\frac{\frac{b+c}{2}+a}{2} = 83.5$. Multiplying by 2 gives $\frac{b+c}{2} + a = 167$. Multiplying by 2 again yields $b + c + 2a = 334$. (2)

When the average of her first and third marks is averaged with her second test mark, the new average is 84.5 so $\frac{\frac{a+c}{2}+b}{2} = 84.5$. Multiplying by 2 gives $\frac{a+c}{2} + b = 169$. Multiplying by 2 again yields $a + c + 2b = 338$. (3)

By adding equations (1), (2) and (3) we obtain $4a + 4b + 4c = 992$, which simplifies to $a + b + c = 248$ after dividing by 4. This means that the sum of her first three marks is 248.

To obtain an average of 86% over the four tests, Erin needs a total of $86 \times 4 = 344$ marks. In other words, $a + b + c + d = 344$. We know that $a + b + c = 248$ so $248 + d = 344$, and $d = 96$ follows.

Therefore, to obtain an average of 86%, Erin needs 96% on her fourth test.

It would be a straight forward process to determine Erin's first three test marks but our question did not ask us to do this. However, for the curious, on her first test Erin got 86, on her second test Erin got 90, and on her third test Erin got 72.

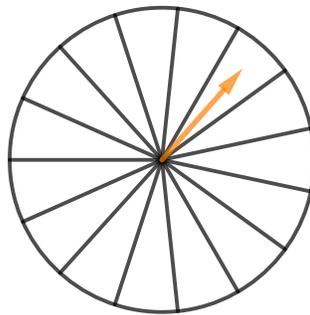


Problem of the Week

Problem E

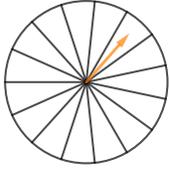
Spin, Spin, Spin

A spinner is divided into 15 equal sections. Each section is coloured either red, green, or yellow. An arrow is attached to the centre of the spinner. Jordan spins the arrow 3 times. If there is a 48.8% chance of landing on red in at least one of the three spins, how many red sections are there?



You may use a known result from probability theory. If the probability of event A occurring is a , the probability of event B occurring is b , the probability of event C occurring is c , and the results are not dependent on each other, then the probability of all three events happening is $a \times b \times c$.





Problem of the Week

Problem E and Solution

Spin, Spin, Spin

Problem

A spinner is divided into 15 equal sections. Each section is coloured either red, green, or yellow. An arrow is attached to the centre of the spinner. Jordan spins the arrow 3 times. If there is a 48.8% chance of landing on red in at least one of the three spins, how many red sections are there?

Solution

There will be two solutions presented. In the second solution, we will need to make use of the Factor Theorem, which is typically taught in a Grade 12 math course. In both we will use the given fact that if the probability of event A occurring is a , the probability of event B occurring is b , the probability of event C occurring is c , and the results are not dependent on each other, then the probability of all three events happening is $a \times b \times c$.

Solution 1:

Let n be the number of non-red sections. Therefore, for each spin, the probability the spinner does not land on a red section is $\frac{n}{15}$. Since the result of each spin does not depend on the previous spin,

$$\begin{aligned} P(\text{no reds in all 3 spins}) &= P(\text{1st not red}) \times P(\text{2nd not red}) \times P(\text{3rd not red}) \\ &= \frac{n}{15} \times \frac{n}{15} \times \frac{n}{15} \\ &= \left(\frac{n}{15}\right)^3 \end{aligned}$$

The probability of landing on at least one red in three spins is 0.488. So, the probability of not landing on red in any of the three spins is $1 - 0.488 = 0.512$.

That is,

$$\begin{aligned} \left(\frac{n}{15}\right)^3 &= 0.512 \\ \left(\frac{n}{15}\right)^3 &= \frac{512}{1000} \\ \frac{n}{15} &= \frac{8}{10} \\ n &= 12 \end{aligned}$$

Since n is the number of non-red sections, there are $15 - 12 = 3$ red sections.

**Solution 2:**

Let r be the number of red sections. Therefore, the number of non-red sections is $15 - r$. Also, for each spin, the probability the spinner lands on a red section is $\frac{r}{15}$ and the probability the spinner does not land on a red section is $\frac{15-r}{15}$.

If Jordan lands on red in at least one spin, then the first red occurs on his first spin, second spin, or third spin.

If Jordan lands on the first red on his first spin, then on his first spin he spins a red, on his second spin he spins any colour, and on his third spin he spins any colour. Since the results of each spin do not depend on each other, the probability that the first red occurs on his first spin is $P(1) = \frac{r}{15} \times \frac{15}{15} \times \frac{15}{15} = \frac{r}{15}$.

If Jordan lands on the first red on the second spin, then on his first spin he did not spin a red, on his second spin he spins a red, and on his third spin he spins any colour. The probability of this is $P(2) = \frac{15-r}{15} \times \frac{r}{15} \times \frac{15}{15} = \frac{(15-r)r}{15^2}$.

If Jordan lands on the first red on the third spin, then on his first two spins he did not spin a red and on this third spin he spins a red. The probability of this is

$$P(3) = \frac{15-r}{15} \times \frac{15-r}{15} \times \frac{r}{15} = \frac{(15-r)^2 r}{15^3}.$$

The probability of getting at least one red in the three spins is equal to $P(1) + P(2) + P(3)$.

That is,

$$\begin{aligned} \frac{r}{15} + \frac{(15-r)r}{15^2} + \frac{(15-r)^2 r}{15^3} &= 0.488 \\ 225r + 15(15-r)r + (15-r)^2 r &= 1647 \\ r^3 - 45r^2 + 675r - 1647 &= 0 \\ (r-3)(r^2 - 42r + 549) &= 0 \quad (*\text{see below to see the factoring}) \end{aligned}$$

Therefore, $r - 3 = 0$ or $r^2 - 42r + 549 = 0$. Using the quadratic formula, we see that $r^2 - 42r + 549 = 0$ has no real solution. Therefore, the only solution is $r = 3$.

That is, there are 3 red sections.

***FACTORING THE CUBIC:**

It is easy to verify the factoring is correct by expanding. To determine the factorization, we used the Factor Theorem. This topic would typically be covered in a Grade 12 Math course.

Let $f(r) = r^3 - 45r^2 + 675r - 1647$. Notice that $f(3) = 0$. Therefore, the Factor Theorem tells us that $(r - 3)$ is a factor of $f(r)$. Therefore, $f(r) = (r - 3)(ar^2 + br + c)$, for some real numbers a, b, c . Expanding this, we get

$$f(r) = (r - 3)(ar^2 + br + c) = ar^3 + br^2 + cr - 3ar^2 - 3br - 3c = ar^3 + (b - 3a)r^2 + (c - 3b)r - 3c.$$

Setting the two expressions for $P(r)$ equal, we get

$$r^3 - 45r^2 + 675r - 1647 = ar^3 + (b - 3a)r^2 + (c - 3b)r - 3c.$$

Setting the coefficients for r^3 equal, we get $a = 1$.

Setting the constants equal, we get $-3c = -1647$ or $c = 549$.

Setting the coefficients for r^2 equal, we get $b - 3a = -45$ or $b = -42$.

Therefore, $f(r) = (r - 3)(r^2 - 42r + 549)$.



Problem of the Week

Problem E

The Power of Pixels

A *pixel* is the smallest unit of a digital image. The number of pixels/cm in each of the horizontal and vertical directions of a digital image affects the quality of the image. The more pixels/cm, the sharper the image is.

A monitor has dimensions 15 cm by 10 cm and has 80 pixels/cm in each dimension. The total number of pixels is $(15 \times 80) \times (10 \times 80) = 960\,000$.

The manufacturer wants to build a new monitor with 2 145 624 pixels. To accomplish this, both the length and width of the screen will be increased by $n\%$ and the number of pixels/cm in each dimension will be increased by $2n\%$.

Determine the dimensions of the new monitor and the new number of pixels/cm.





Problem of the Week

Problem E and Solution

The Power of Pixels

Problem

A *pixel* is the smallest unit of a digital image. The number of pixels/cm in each of the horizontal and vertical directions of a digital image affects the quality of the image. The more pixels/cm, the sharper the image is. A monitor has dimensions 15 cm by 10 cm and has 80 pixels/cm in each dimension. The total number of pixels is $(15 \times 80) \times (10 \times 80) = 960\,000$. The manufacturer wants to build a new monitor with 2 145 624 pixels. To accomplish this, both the length and width of the screen will be increased by $n\%$ and the number of pixels/cm in each dimension will be increased by $2n\%$. Determine the dimensions of the new monitor and the new number of pixels/cm.

Solution

Let $a > 0$ represent the percentage increase, expressed as a decimal, in each dimension of the monitor. Then $2a$ represents the percentage increase, expressed as a decimal, in the number of pixels/cm. (So $n = 100a$ and $2n = 200a$.)

We can write an equation to represent the total number of pixels.

$$[(\text{New Length}) \times (\text{New pixels/cm})] \times [(\text{New Width}) \times (\text{New pixels/cm})] = \text{Total Number of Pixels}$$

$$[15(1 + a) \times 80(1 + 2a)] \times [10(1 + a) \times 80(1 + 2a)] = 2\,145\,624$$

$$(15)(80)(10)(80)(1 + a)^2(1 + 2a)^2 = 2\,145\,624$$

$$960\,000(1 + a)^2(1 + 2a)^2 = 2\,145\,624$$

$$\text{Divide both sides by } 960\,000: \quad (1 + a)^2(1 + 2a)^2 = 2.235\,025$$

$$\text{Take the square root of both sides:} \quad (1 + a)(1 + 2a) = \pm 1.495$$

$$\text{Since } a > 0, 1 + a > 0, 1 + 2a > 0 \text{ and } (1 + a)(1 + 2a) > 0,$$

$$-1.495 \text{ is inadmissible, so:} \quad (1 + a)(1 + 2a) = 1.495$$

$$2a^2 + 3a + 1 = 1.495$$

$$2a^2 + 3a - 0.495 = 0$$

Using the quadratic formula:

$$a = \frac{-3 \pm \sqrt{9 - 4(2)(-0.495)}}{4}$$

$$a = \frac{-3 \pm \sqrt{12.96}}{4}$$

$$a = \frac{-3 \pm 3.6}{4}$$

It follows that $a = 0.15$ or $a = -1.65$. But we are looking for a percentage increase so $a > 0$ and $a = -1.65$ is inadmissible. The dimensions of the screen must each be increased by 15% and the numbers of pixels/cm must be increased by $2 \times 15\% = 30\%$.

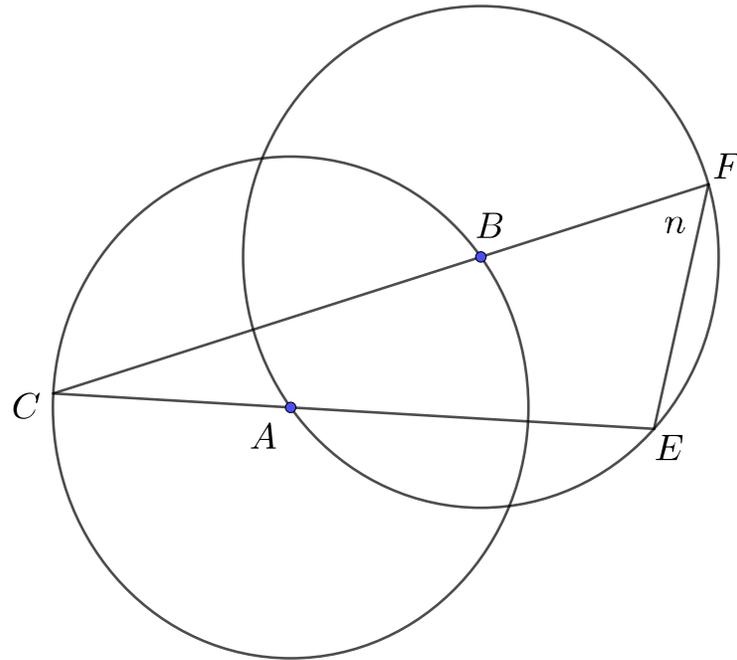
The increased length is $15 \times 1.15 = 17.25$ cm and the increased width is $10 \times 1.15 = 11.5$ cm. The increased number of pixels/cm is $80 \times 1.3 = 104$ pixels/cm.



Problem of the Week

Problem E

The Angle Chase is On



Two intersecting circles are constructed so that the centre of one circle is on the circumference of the other circle. That is, A is the centre of a circle which passes through B and B is the centre of a circle which passes through A . CAE and CBF are straight line segments.

If $\angle F = n$, with $0^\circ < n < 90^\circ$, determine the measure of $\angle C$ in terms of n .





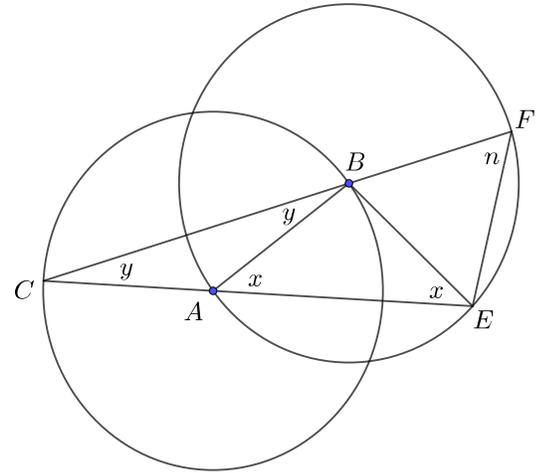
Problem of the Week

Problem E and Solution

The Angle Chase is On

Problem

Two intersecting circles are constructed so that the centre of one circle is on the circumference of the other circle. That is, A is the centre of a circle which passes through B and B is the centre of a circle which passes through A . CAE and CBF are straight line segments. If $\angle F = n$, with $0^\circ < n < 90^\circ$, determine the measure of $\angle C$ in terms of n .



Solution

Construct AB and BE . A , E and F are on the circumference of the circle with centre B . Therefore, $BA = BE = BF$.

In $\triangle BEF$, $BE = BF$. Then $\triangle BEF$ is isosceles and $\angle BEF = \angle F = n$.

Let $\angle BEA = x$ and $\angle ABC = y$.

In $\triangle BAE$, $BA = BE$ and the triangle is isosceles. Therefore, $\angle BAE = \angle BEA = x$.

Since B and C are on the circle with centre A , $AC = AB$ and $\triangle ABC$ is isosceles. Therefore, $\angle C = \angle ABC = y$.

$\angle EAB$ is an exterior angle to $\triangle ABC$. By the exterior angle theorem for triangles, $\angle EAB = \angle C + \angle ABC$. But $\angle EAB = x$ and $\angle C + \angle ABC = y + y = 2y$. Therefore, $x = 2y$.

In $\triangle CEF$,

$$\angle C + \angle E + \angle F = 180^\circ$$

$$\angle C + \angle CEB + \angle BEF + n = 180^\circ$$

$$y + x + n + n = 180^\circ$$

Also, $x = 2y$. Therefore,

$$y + 2y + 2n = 180^\circ$$

$$3y = 180^\circ - 2n$$

$$y = \frac{180^\circ - 2n}{3}$$

$$y = 60^\circ - \frac{2}{3}n$$

Therefore, $\angle C = 60^\circ - \frac{2}{3}n$.



Problem of the Week

Problem E

An Equal Distribution?

A group of workers won a cash award. All of the money was to be distributed among the group members in the following way:

$\$x$ to the oldest member of the group plus $\frac{1}{16}$ of what remains, then

$\$2x$ to the second oldest member of the group plus $\frac{1}{16}$ of what then remains, then

$\$3x$ to the third oldest member of the group plus $\frac{1}{16}$ of what then remains, and so on.

When the distribution of the money was complete, each group member received the same amount and no money was left over. Determine the number of members in the group.





Problem of the Week

Problem E and Solution

An Equal Distribution?

Problem

A group of workers won a cash award. All of the money was to be distributed among the group members in the following way:

- $\$x$ to the oldest member of the group plus $\frac{1}{16}$ of what remains, then
- $\$2x$ to the second oldest member of the group plus $\frac{1}{16}$ of what then remains, then
- $\$3x$ to the third oldest member of the group plus $\frac{1}{16}$ of what then remains, and so on.

When the distribution of the money was complete, each group member received the same amount and no money was left over. Determine the number of members in the group.

Solution

Let T be the total value of the cash award. Since there is a cash award, $T > 0$.

Let y be the amount of money given to each group member. Since each group member receives money, $y > 0$.

Then $\frac{T}{y}$ is the number of group members.

The oldest group member receives x to begin with. There would be $(T - x)$ left at this point. The oldest group member then receives $\frac{1}{16}$ of the remaining amount $(T - x)$. Therefore, the oldest group member receives $y = x + \frac{1}{16}(T - x)$.

The second oldest group member receives $2x$ to begin with. There would now be $(T - y - 2x)$ left at this point. This represents the original amount minus the oldest member's full share minus the amount received so far by the second oldest group member. The second oldest group member then receives $\frac{1}{16}$ of the remaining amount $(T - y - 2x)$. Therefore, the second oldest group member receives $y = 2x + \frac{1}{16}(T - y - 2x)$.

(The solution continues on the next page.)



But each group member receives the same amount. Since $y = x + \frac{1}{16}(T - x)$ and $y = 2x + \frac{1}{16}(T - y - 2x)$ it follows that

$$\begin{aligned}x + \frac{1}{16}(T - x) &= 2x + \frac{1}{16}(T - y - 2x) \\x + \frac{1}{16}T - \frac{1}{16}x &= 2x + \frac{1}{16}T - \frac{1}{16}y - \frac{1}{16}(2x) \\x - \frac{1}{16}x &= 2x - \frac{1}{16}y - \frac{2}{16}x\end{aligned}$$

Multiply both sides by 16:

$$\begin{aligned}16x - x &= 32x - y - 2x \\15x &= 30x - y \\y &= 15x\end{aligned}$$

Therefore each group member receives $\$15x$.

Substituting $15x$ for y into the equation $y = x + \frac{1}{16}(T - x)$ we obtain

$$\begin{aligned}x + \frac{1}{16}(T - x) &= 15x \\ \frac{1}{16}(T - x) &= 14x \\ T - x &= 224x \\ T &= 225x\end{aligned}$$

Therefore the total value of the cash award is $\$225x$.

We can now determine the number of group members $\frac{T}{y} = \frac{225x}{15x} = 15$.

Therefore, there were 15 group members.

Notice, we did not need to know T , the total value of the cash award. It turns out that once we know x , we can determine each group member's share, $\$15x$, and the total value of the award, $T = \$225x$.



Problem of the Week

Problem E

Yes We Can!

A local food bank has created a unique 100-day plan for collecting canned food donations.

Day 1 Goal: Collect 50 cans of food.

Day 2 Goal: Collect 3 more cans of food than the current day number plus the same number of cans collected on day 1.

Day 3 Goal: Collect 3 more cans of food than the current day number plus the same number of cans collected on day 2.

Day 4 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 3.

⋮

Day 100 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 99.

How many cans of food will the food bank collect on the 100th day?



Did you know that the sum of the positive integers from 1 to n can be determined using the formula $\frac{n(n+1)}{2}$? For example, the sum of the integers $1 + 2 + 3 + 4 = \frac{4(5)}{2} = 10$. This result can be verified by simply adding the 4 numbers. You can also easily verify that the sum of the first 5 positive integers is $\frac{5(6)}{2} = 15$.

Depending on your approach to the problem, this formula may be useful. As a challenge, one may wish to prove this formula holds for any positive integer n .

Extension: Assuming their target is met each day of the 100-day campaign, how many cans of food will they collect in total?





Problem of the Week

Problem E and Solution

Yes We Can!

Problem

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Day 4 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 3.

⋮

Day 100 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 99.

How many cans of food will the food bank collect on the 100th day?

Extension: Assuming their target is met each day of the 100-day campaign, how many cans of food will they collect in total?

Solution

First we will introduce function notation to represent the information in the problem.

Let n represent the day number and $f(n)$ represent the number of cans collected on day n .

We know that on the first day, 50 cans were collected. So, $f(1) = 50$.

On the second day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 1. So, $f(2) = 3 + 2 + f(1) = 3 + 2 + 50 = 55$.

On the third day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 2. So, $f(3) = 3 + 3 + f(2) = 3 + 3 + 55 = 61$.

On the fourth day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 3. So, $f(4) = 3 + 4 + f(3) = 3 + 4 + 61 = 68$.

On the n^{th} day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day $(n - 1)$. So, $f(n) = 3 + n + f(n - 1)$.

This sequence of cans collected can be defined recursively as follows:

$$f(n) = \begin{cases} 50, & \text{if } n = 1. \\ f(n - 1) + n + 3, & n \geq 2, n \in \mathbb{Z}. \end{cases}$$

We want to find $f(100)$, the 100th term in the sequence, and if we complete the extension, we want $f(1) + f(2) + f(3) + \cdots + f(100)$. Several approaches follow on the next pages.



Approach 1

One obvious, yet exhausting, approach to determining the value collected on the 100th day would be to determine the values of all the preceding days. This is not really practical and will not be pursued here.

Approach 2

Earlier we noted that $f(1) = 50$, $f(2) = 55$, $f(3) = 61$, and $f(4) = 68$. From $f(1)$ to $f(2)$ the sequence of cans collected increased by 5, from $f(2)$ to $f(3)$ the sequence increased by 6, and from $f(3)$ to $f(4)$ the sequence increased by 7. We see a pattern and might predict that going from $f(4)$ to $f(5)$ the sequence would increase by 8. Calculating $f(5)$ using the function we get $f(5) = f(4) + 5 + 3 = 68 + 8 = 76$. Our prediction was correct but does it generalize?

To justify this further, if $f(p-1)$ and $f(p)$ are adjacent terms in the sequence of cans collected, then $f(p) = f(p-1) + p + 3$. Rearranging, $f(p) - f(p-1) = p + 3$. That is, the difference between consecutive terms will be the term number of the term in the higher position plus 3. So $f(6) - f(5) = 6 + 3 = 9$ and $f(7) - f(6) = 7 + 3 = 10$.

Can we use this to get from the first term, $f(1)$, to the the hundredth term, $f(100)$? Going from the first term to the hundredth term we advance 99 terms. The value of the hundredth term would be 50, the value of the first term, plus 99 consecutive integers starting with 5.

That sum is $50 + (5 + 6 + 7 + 8 + \cdots + 100 + 101 + 102 + 103)$

$$= 50 + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \cdots + 100 + 101 + 102 + 103) - (1 + 2 + 3 + 4).$$

Using the formula $\frac{n(n+1)}{2}$ with $n = 103$ and $n = 4$, this becomes

$$50 + \frac{103(104)}{2} - \frac{4(5)}{2} = 50 + 5356 - 10 = 5396.$$

That is, 5396 cans are collected on the 100th day.



Approach 3

The content in this approach may be unfamiliar to many of you reading it.

From our work so far, we create a table:

Day Number	1	2	3	4	5	6
Value	50	55	61	68	76	85

If we treat the day number as the independent variable, say x , and the value as the dependent variable, say y , we can make some conclusions. First, Δx is constant. However, Δy , the first differences are not constant. Therefore, the function is not linear.

The first differences of the given values are 5, 6, 7, 8, and 9. If we calculate the second differences, each of them is 1. That is, for a constant Δx , the first differences are not constant but the second differences are constant. This tells us that the function is quadratic, that is, $f(x) = ax^2 + bx + c, x \geq 1, x \in \mathbb{Z}$. The symbol \mathbb{Z} is used for the set of integers.

We know that $f(1) = a(1)^2 + b(1) + c = a + b + c$ and $f(1) = 50$, so

$$a + b + c = 50 \quad (1)$$

We know that $f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$ and $f(2) = 55$, so

$$4a + 2b + c = 55 \quad (2)$$

We know that $f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$ and $f(3) = 61$, so

$$9a + 3b + c = 61 \quad (3)$$

We will eliminate c from the system of equations.

$$(2) - (1) \quad 3a + b = 5 \quad (4)$$

$$(3) - (2) \quad 5a + b = 6 \quad (5)$$

Now, eliminate b from the system of equations.

$$(5) - (4) \quad 2a = 1 \Rightarrow a = \frac{1}{2}$$

Substitute in (4) to find b : $3\left(\frac{1}{2}\right) + b = 5 \Rightarrow b = 5 - \frac{3}{2} = \frac{7}{2}$

Substitute in (1) to find c : $\frac{1}{2} + \frac{7}{2} + c = 50 \Rightarrow c = 50 - 4 = 46$

Therefore, the quadratic function is $f(x) = \frac{1}{2}x^2 + \frac{7}{2}x + 46, x \geq 1, x \in \mathbb{Z}$.

To determine the value collected on the 100th day, evaluate $f(x)$ for $x = 100$.

$$f(100) = \frac{1}{2}(100)^2 + \frac{7}{2}(100) + 46 = 5000 + 350 + 46 = 5396$$

That is, 5396 cans are collected on the 100th day.

Using this approach, we can find the amount collected on any day, since we have a general formula for the value collected given the day number. That is, the amount collected on day n is $f(n) = \frac{1}{2}n^2 + \frac{7}{2}n + 46$, where $n \geq 1, n \in \mathbb{Z}$.



Approach 4

The content in this approach will probably be unfamiliar to most of you reading it.

We are given this following definition:

$$f(n) = \begin{cases} 50, & \text{if } n = 1. \\ f(n-1) + n + 3, & n \geq 2, n \in \mathbb{Z}. \end{cases}$$

Rearranging the definition of the function for $n \geq 2$ we obtain:

$$f(n) - f(n-1) = n + 3$$

Using $f(n) - f(n-1) = n + 3$ with different values of n :

$$\begin{array}{ll} \text{When } n = 100, & f(100) - f(99) = 100 + 3 \\ \text{When } n = 99, & f(99) - f(98) = 99 + 3 \\ \text{When } n = 98, & f(98) - f(97) = 98 + 3 \\ & \vdots \\ \text{When } n = 4, & f(4) - f(3) = 4 + 3 \\ \text{When } n = 3, & f(3) - f(2) = 3 + 3 \\ \text{When } n = 2, & \underline{f(2) - f(1) = 2 + 3} \end{array}$$

If we add all of the terms on the left side of each equal sign, we are left with only $f(100) - f(1)$ since we have $-f(99) + f(99)$, $-f(98) + f(98)$, \dots , $-f(3) + f(3)$, and $-f(2) + f(2)$.

If we add all of the terms on the right side of each equal sign, we get

$$2 + 3 + 4 + \dots + 98 + 99 + 100 + 99(3)$$

$$\text{Therefore, } f(100) - f(1) = 2 + 3 + 4 + \dots + 98 + 99 + 100 + 99(3) \quad (*)$$

$$\begin{aligned} \text{Since } f(1) = 50, \quad f(100) - 50 &= 5049 + 297 \\ f(100) &= 5049 + 297 + 50 \\ &= 5396 \end{aligned}$$

That is, 5396 cans are collected on the 100th day.

$$\begin{aligned} \text{Note concerning } (*) \text{ above: } & 2 + 3 + 4 + \dots + 98 + 99 + 100 \\ &= 1 + 2 + 3 + 4 + \dots + 98 + 99 + 100 - 1 \\ &= \frac{100(101)}{2} - 1 \\ &= 5050 - 1 \\ &= 5049 \end{aligned}$$



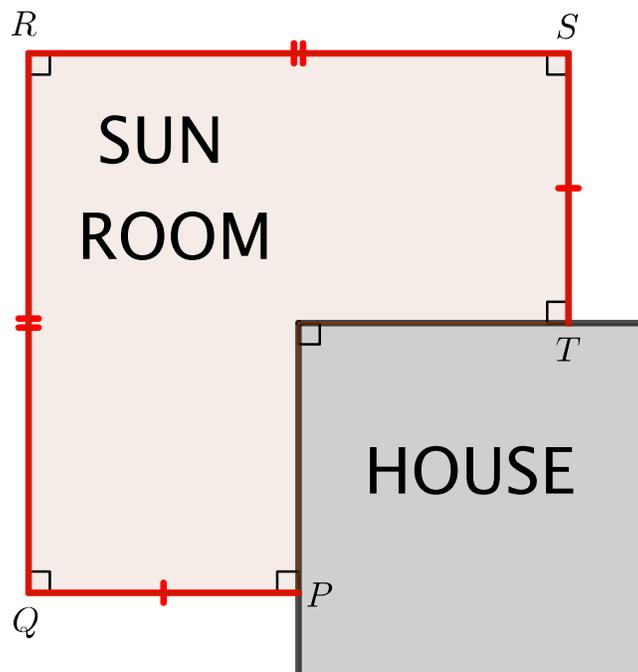
Problem of the Week

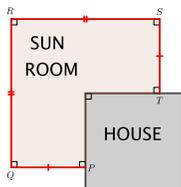
Problem E

Adding On

A sun room is to be built out from one of the corners of a house, as illustrated below. The lengths of the two longer outer walls, RS and QR , are to be the same, and the lengths of the two shorter outer walls, ST and PQ , are to be the same. These walls will be at right angles to each other and to the existing house. The total distance from P to Q to R to S to T is to be 30 m. Determine the dimensions that will give the maximum sun room area.

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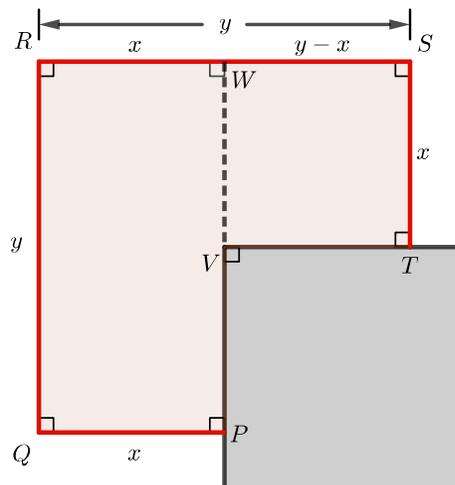
Problem of the Week

Problem E and Solution

Adding On

Problem

A sun room is to be built out from one of the corners of a house, as illustrated above. The lengths of the two longer outer walls, RS and QR , are to be the same, and the lengths of the two shorter outer walls, ST and PQ , are to be the same. These walls will be at right angles to each other and to the existing house. The total distance from P to Q to R to S to T is to be 30 m. Determine the dimensions that will give the maximum sun room area.



Solution

Label the corner of the house V . Draw a line segment through PV to RS intersecting at W . Then $PW \perp RS$ and this makes two rectangles, $PQRW$ and $WSTV$.

Let x represent the lengths of both PQ and ST . Let y represent the lengths of both QR and RS . Since $PQRW$ is a rectangle, $RW = PQ = x$ and $WS = RS - RW = y - x$.

Let A represent the area of the sun room.

The total distance from P to Q to R to S to T is

$$PQ + QR + RS + ST = x + y + y + x = 2x + 2y = 30$$

Dividing by 2, we get $x + y = 15$. Rearranging to solve for y , we obtain $y = 15 - x$. (1)

$$\begin{aligned} \text{Area of sun room} &= \text{Area } PQRW + \text{Area } WSTV \\ &= QR \times RW + WS \times ST \\ &= yx + (y - x)x \\ &= (15 - x)x + ((15 - x) - x)x, \quad \text{substituting from (1) above} \\ &= (15 - x)x + (15 - 2x)x \\ &= 15x - x^2 + 15x - 2x^2 \\ &= -3x^2 + 30x \\ &= -3(x^2 - 10x) \\ &= -3(x^2 - 10x + 5^2 - 5^2), \quad \text{by completing the square} \\ &= -3(x^2 - 10x + 25) + 75 \end{aligned}$$

$$\therefore A = -3(x - 5)^2 + 75$$

This is the equation of a parabola which opens down from a vertex of $(5, 75)$. The maximum area is 75 m^2 when $x = 5$ m. When $x = 5$, $y = 15 - x = 15 - 5 = 10$ m.

Therefore, if the longer side of the sun room is 10 m and the shorter side of the sun room is 5 m, this gives a maximum area of 75 m^2 .



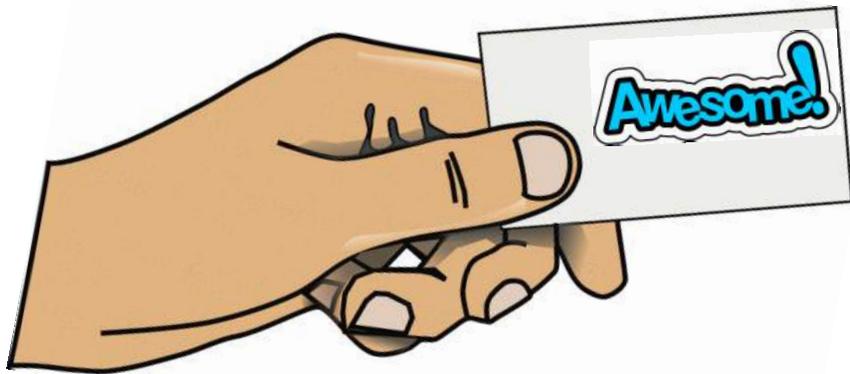
Problem of the Week

Problem E

An Encouraging Word

Every student in a class at Sunshine High School committed to sending a card to every other student in the class with some word of encouragement on it. The students that have a first name beginning with a letter from A to K received a total of 459 cards. The remaining students that have a first name beginning with a letter from L to Z received a total of 297 cards.

How many students are in this class? How many students in the class have a first name beginning with a letter from A to K?





Problem of the Week
Problem E and Solution
An Encouraging Word



Problem

Every student in a class at Sunshine High School committed to sending a card to every other student in the class with some word of encouragement on it. The students that have a first name beginning with a letter from A to K received a total of 459 cards. The remaining students that have a first name beginning with a letter from L to Z received a total of 297 cards. How many students are in this class? How many students in the class have a first name beginning with a letter from A to K?

Solution

Let n represent the number of students in the class.

Each student sends $n - 1$ notes and the total number of notes sent is $459 + 297$ or 756. Also, the number of students, n , times the number of notes sent by each student, $n - 1$, equals the total number of notes sent. That is,

$$\begin{aligned}n(n - 1) &= 756 \\n^2 - n - 756 &= 0 \\(n - 28)(n + 27) &= 0 \\n = 28 \quad n = -27\end{aligned}$$

Since n represents the number of students in the class, n is greater than 0 and it follows that $n = -27$ is inadmissible. Therefore, there are 28 students in the class and each student sent 27 notes and each received 27 notes.

Since the students with a first name beginning with a letter from A to K received a total of 459 notes and each student received 27 notes, there are $459 \div 27 = 17$ students in the class whose first name begins with a letter from A to K.

The class has 28 students, and 17 of these students have a first name that begins with a letter from A to K.

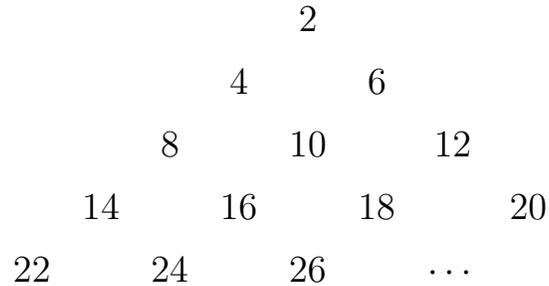


Problem of the Week

Problem E

This is the Year 2

The positive even integers are arranged in increasing order in a triangle, as shown:



Each row contains one more number than the previous row. One of the rows of the triangle contains the number 2020. Determine the sum of the numbers in the row that contains 2020.



Did you know?

There is a quick way to calculate the sum $1 + 2 + 3 + 4 + \dots + 19 + 20$?

$$1 + 2 + 3 + 4 + \dots + 19 + 20 = \frac{(20)(20 + 1)}{2} = 210$$

In general, it can be shown that if n is a positive integer, then the sum of the integers from 1 to n is $S = 1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$.



Problem of the Week

Problem E and Solution

This is the Year 2

Problem

The positive even integers are arranged in increasing order in a triangle, as shown:

			2			
		4		6		
	8		10		12	
14		16		18		20
22	24		26		...	

Each row contains one more number than the previous row. One of the rows of the triangle contains the number 2020. Determine the sum of the numbers in the row that contains 2020.

Solution

To begin the solution, we make an observation: the row number in the triangle is the same as the number of numbers in that particular row. For example, row 1 contains 1 number, row 2 contains 2 numbers, and row r contains r numbers. If we add up all of the row numbers from row 1 to row r we get the number of numbers in the table and the largest number in the row is $2r$.

For example, if we calculate the sum $1 + 2 + 3 + 4$, we get 10. This number corresponds to the number of numbers in the first four rows of the triangle and the largest number in the fourth row is $2 \times 10 = 20$.

Let n represent the row number of the row that contains the number 2020. Since $2020 = 2 \times 1010$ we need to find the row where the 1010th number occurs.

Using the formula for the sum of the integers from 1 to n , we want $\frac{n(n+1)}{2} \geq 1010$. Multiplying by 2, the equation simplifies to $n(n+1) \geq 2020$.

We will use trial and error to find the possible value of n . Finding the $\sqrt{2020}$ will get us a good place to start checking values of n . ($\sqrt{2020} \approx 44.9$.)

When $n = 44$, $44 \times 45 = 1980 < 2020$ and $\frac{44 \times 45}{2} = 990$.

When $n = 45$, $45 \times 46 = 2070 > 2020$ and $\frac{45 \times 46}{2} = 1035$.

These guesses are quite useful. When there are 44 rows in the triangle, the last number in the 44th row is $990 \times 2 = 1980$. So the first number in the 45th row is 1982. When there are 45 rows in the triangle, the last number in the 45th row is $2 \times 1035 = 2070$. We want the sum of the numbers in the 45th row. To do this we will add all of the even integers from 2 to 2020. But this sum includes extra even integers from 2 to 1980. We will use our formula to compute this second sum and subtract it from the first sum to obtain the desired result.



$$\begin{aligned}\text{sum} &= 1982 + 1984 + 1986 + \cdots + 2066 + 2068 + 2070 \\ &= (\mathbf{2 + 4 + 6 + \cdots + 1976 + 1978 + 1980} + 1982 + 1984 + 1986 + \cdots + 2066 + 2068 + 2070) \\ &\quad - (\mathbf{2 + 4 + 6 + \cdots + 1976 + 1978 + 1980}) \\ &= (2(\mathbf{1 + 2 + 3 + \cdots + 988 + 989 + 990} + 991 + 992 + 993 + \cdots + 1033 + 1034 + 1035)) \\ &\quad - (2(\mathbf{1 + 2 + 3 + \cdots + 988 + 989 + 990})) \\ &= 2 \times \left(\frac{1035 \times 1036}{2} \right) - 2 \times \left(\frac{990 \times 991}{2} \right) \\ &= 91\,170\end{aligned}$$

The sum of the numbers in the row containing 2020 is 91 170.



Problem of the Week

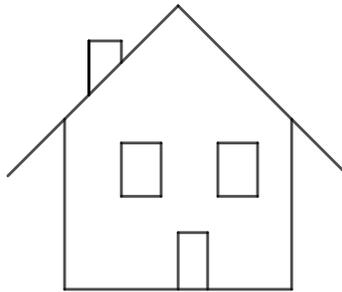
Problem E

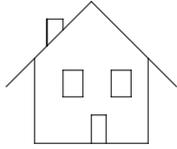
Square Neighbours

Alyssa, Bilal, Constantine and Daiyu all live on the same street.

A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Constantine's house number is 31 more than Alyssa's house number and Daiyu's house number is 31 more than Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number.

What is the house number of each person?





Problem of the Week

Problem E and Solution

Square Neighbours

Problem

Alyssa, Bilal, Constantine and Daiyu all live on the same street. A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Constantine's house number is 31 more than Alyssa's house number and Daiyu's house number is 31 more than Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number. What is the house number of each person?

Solution

Both Alyssa's house number and Bilal's house number must be two-digit numbers. If Alyssa's house number is a one-digit number, Bilal's house number would have to be a three-digit number to create the four-digit perfect square. This means that Constantine's house number would be a two-digit number and Daiyu's house number would be at least a three-digit number and when you combine these two numbers it will be at least a five-digit number. A similar argument can be presented if Bilal's house number is a one-digit number. Therefore, both Alyssa and Bilal have house numbers that are each two-digit numbers.

Let Alyssa's house number be x and Bilal's house number be y . Then $100x + y$ is the four digit number created by writing Alyssa's house number followed by Bilal's house number. But $100x + y$ is a perfect square, so let $100x + y = k^2$, for some positive integer k . (1)

Therefore, Constantine's house number is $(x + 31)$ and Daiyu's house number is $(y + 31)$. The new number created by writing Constantine's house number followed by Daiyu's house number is $100(x + 31) + (y + 31)$. This new four-digit number is also a perfect square. So $100(x + 31) + (y + 31) = m^2$, for some positive integer m , with $m > k$. This simplifies as follows:

$$\begin{aligned}100x + 3100 + y + 31 &= m^2 \\100x + y + 3131 &= m^2\end{aligned}\quad (2)$$



From (1) above, we have $100x + y = k^2$. Substituting k^2 for $100x + y$ in (2) we obtain $k^2 + 3131 = m^2$ or $3131 = m^2 - k^2$. We observe that $m^2 - k^2$ is a difference of squares, so $m^2 - k^2 = (m + k)(m - k) = 3131$.

Since m and k are positive integers, $m + k$ is a positive integer and $m + k > m - k$. Also, $m - k$ must be a positive integer since $(m + k)(m - k) = 3131$. So we are looking for two positive integers that multiply to 3131. There are two possibilities, 3131×1 or 101×31 .

First we will examine $(m + k)(m - k) = 3131 \times 1$. From this we obtain two equations in two unknowns, namely $m + k = 3131$ and $m - k = 1$. Subtracting the two equations gives $2k = 3130$ or $k = 1565$. Then $k^2 = 1565^2 = 2\,449\,225$. This is not a four-digit number, so 3131×1 is not an admissible factorization of 3131.

Next we examine $(m + k)(m - k) = 101 \times 31$. This leads to $m + k = 101$ and $m - k = 31$. Subtracting the two equations we get $2k = 70$ or $k = 35$. Then $100x + y = k^2 = 1225$. Therefore, $x = 12$ and $y = 25$, since 1225 is the four-digit number formed by writing Alyssa's house number, x , followed by Bilal's house number, y .

Now, Constantine's house number $x + 31 = 12 + 31 = 43$ and Daiyu's house number is $y + 31 = 25 + 31 = 56$. Notice that $4356 = 66^2$, so it is a four-digit perfect square.

Therefore, Alyssa's house number is 12, Bilal's house number is 25, Constantine's house number is 43, and Daiyu's house number is 56.



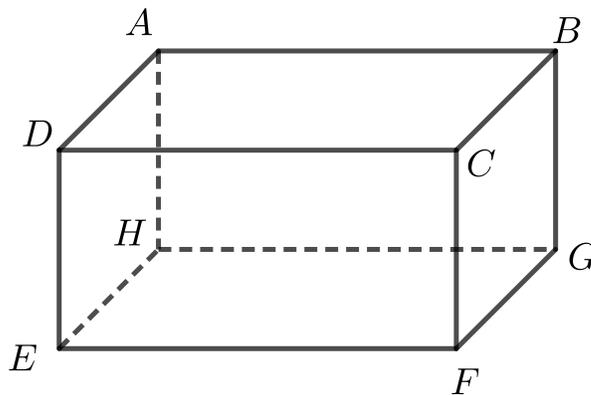
Problem of the Week

Problem E

Body Diagonal

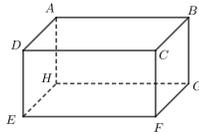
In the given rectangular prism $ABCDEFGH$,

- the sum of the lengths of all of the edges is 28 cm, and
- the total surface area is 13 cm^2 .



What is the length of the diagonal EB ?





Problem of the Week

Problem E and Solution

Body Diagonal

Problem

In the above rectangular prism $ABCDEFGH$,

- the sum of the lengths of all of the edges is 28 cm, and
- the total surface area is 13 cm^2 .

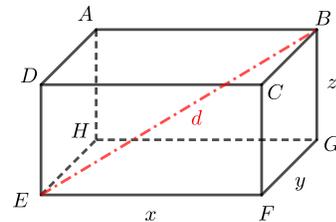
What is the length of the diagonal EB ?

Solution

Let $EF = x$, $FG = y$ and $BG = z$.

We construct EB and label it d .

The updated diagram is shown to the right.



By the Pythagorean Theorem in $\triangle EBG$, $EB^2 = EG^2 + BG^2$.

By the Pythagorean Theorem in $\triangle EFG$, $EG^2 = EF^2 + FG^2$.

Therefore, $EB^2 = EG^2 + BG^2 = EF^2 + FG^2 + BG^2$.

That is, $d^2 = x^2 + y^2 + z^2$.

Since the sum of the lengths of all the edges is 28, then $4x + 4y + 4z = 28$ or

$$x + y + z = 7.$$

Since the surface area of the prism is 13, we know $2xy + 2yz + 2xz = 13$.

Since we have squared terms and pair factor terms it might be helpful to expand $(x + y + z)^2$.

$$\begin{aligned} (x + y + z)^2 &= (x + (y + z))^2 \\ &= x^2 + 2x(y + z) + (y + z)^2 \\ &= x^2 + 2xy + 2xz + y^2 + 2xy + z^2 \end{aligned}$$

$$\text{Therefore, } \underbrace{(x + y + z)^2}_{7^2} = \underbrace{x^2 + y^2 + z^2}_{d^2} + \underbrace{2xy + 2xz + 2yz}_{13}$$

Substituting what we know,

$$49 - 13 = d^2$$

$$d^2 = 36$$

$$d = 6, \text{ since } d > 0$$

Therefore, the length of EB is 6 cm.



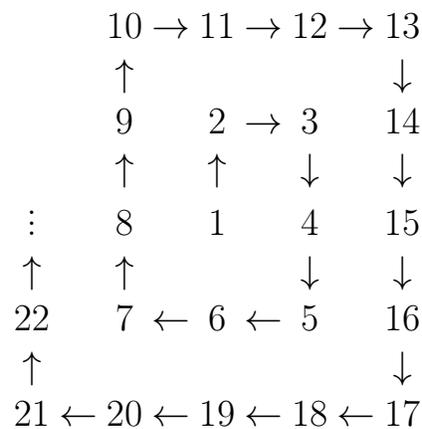
Problem of the Week

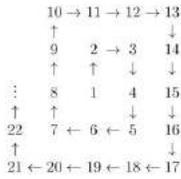
Problem E

Spiral

A spiral of numbers is created, as shown, starting with 1.

If the pattern of the spiral continues, how will the numbers 2020, 2021, and 2022 appear in the spiral? (Will they appear left to right in a row? Right to left in a row? Down in a column? Up in a column? Down and then left? Up and then right?)





Problem of the Week

Problem E and Solution

Spiral

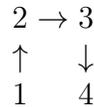
Problem

A spiral of numbers is created, as shown above, starting with 1. If the pattern of the spiral continues, how will the numbers 2020, 2021, and 2022 appear in the spiral? (Will they appear left to right in a row? Right to left in a row? Down in a column? Up in a column? Down and then left? Up and then right?)

Solution

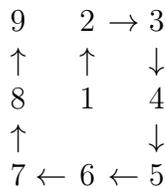
We are looking for a pattern which can be used to predict the positions of the numbers 2020, 2021, and 2022.

Observe a first “square” in the middle containing the numbers 1, 2, 3, and 4, with 4 in the lower right corner.



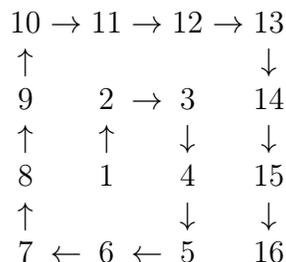
Starting with the 1, to get this “square” we need to add one number above and one number to right. We also need to add a number that is in the diagonal up and to the right. That is, we need to add three numbers to get the first “square”.

Extend the diagram to create a second “square”:



To get this second “square”, we need to add two numbers below the first “square” and two numbers to the left of the first “square”. We also need to add a number that is in the diagonal down and to the left. That is, we need to add five numbers to get from the first “square” to the second “square”.

Extend the diagram to create a third “square”:

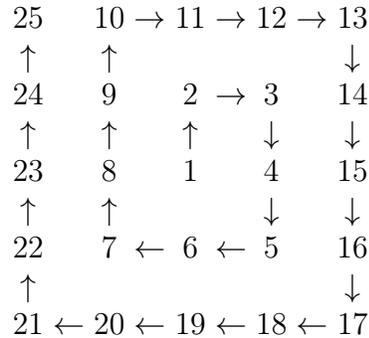


To get this third “square”, we need to add three numbers above the second “square” and three numbers to the right of the second “square”. We also need to add a number that is in the



diagonal up and to the right. That is, we need to add seven numbers to get from the second “square” to the third “square”.

Extend the diagram to create a fourth “square”:



To get this fourth “square” we need to add four numbers below the third “square” and four numbers to the left of the third “square”. We also need to add a number that is in the diagonal down and to the left. That is, we need to add nine numbers to get from the third “square” to the fourth “square”.

If this pattern continues, the fifth “square” would ‘end’ with $25 + 11 = 36$ in the bottom right position. The sixth “square” would end with $36 + 13 = 49$ in the top left. Notice that the numbers 4, 9, 16, 25, 36, and 49 are all perfect squares. Also notice that the even perfect squares are in the bottom right corners and the odd perfect squares are in the top left corners. Now, 2020 lies between $44^2 = 1936$ and $45^2 = 2025$. Since 2025 is an odd perfect square, 2025 will be in the top left corner of a “square”, and the left side of the “square” will look like this:



Therefore, 2020, 2021 and 2022 will be going up in a column.



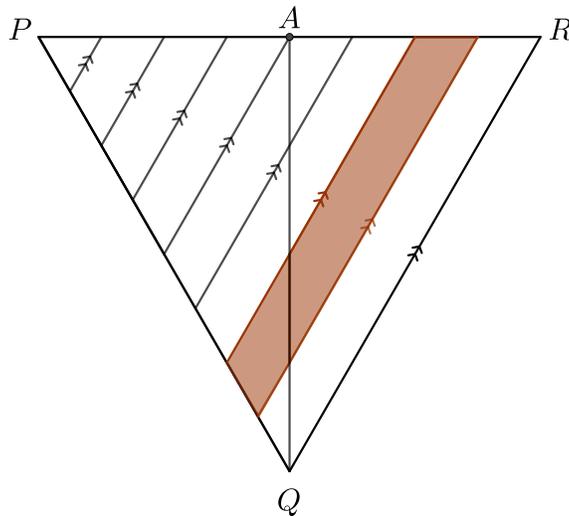
Problem of the Week

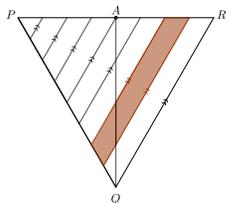
Problem E

Shady Division

$\triangle PQR$ is an equilateral triangle with sides of length 16 cm. Two sides of the triangle, PR and PQ , are each divided into 8 segments of equal length. Each point of division on PR is connected to its corresponding point of division on PQ , creating 7 line segments. The region formed between two of these line segments is shaded, as shown.

An altitude is constructed from Q to A on PR , dividing the shaded region into two parts. In this shaded region, determine the ratio of the area on the left side of the altitude to the area on the right side of the altitude.





Problem of the Week

Problem E and Solution

Shady Division

Problem

$\triangle PQR$ is an equilateral triangle with sides of length 16 cm. Two sides of the triangle, PR and PQ , are each divided into 8 segments of equal length. Each point of division on PR is connected to its corresponding point of division on PQ , creating 7 line segments. The region formed between two of these line segments is shaded, as shown. An altitude is constructed from Q to A on PR , dividing the shaded region into two parts. In this shaded region, determine the ratio of the area on the left side of the altitude to the area on the right side of the altitude.

Solution

Solution 1:

Since PR and PQ are each divided into 8 equal segments, then each segment will have a length of $16 \div 8 = 2$ cm.

Label the shaded region $DEFG$. Let B and C be where altitude QA intersects $DEFG$, as shown to the right. We are required to find the ratio of the area of $DBCG$ to the area of $CBEF$. We will find the area of $DEFG$ by finding the area of $\triangle DPE$ and subtracting it from the area of $\triangle GPF$.

We first mention a few facts that we will use in our solution. For some equilateral $\triangle WXY$ with side length 2, altitude WZ right bisects base XY and it follows that $XZ = ZY = 1$. Using the Pythagorean Theorem, the length of WZ is $\sqrt{2^2 - 1^2} = \sqrt{3}$. In any equilateral triangle, the ratio of the height to the side length is $\sqrt{3} : 2$. In other words, the height of any equilateral triangle is $\frac{\sqrt{3}}{2}$ times its side length. The altitude WZ also bisects $\angle XWY$, so $\angle XWZ = \angle YWZ = 30^\circ$ and $\triangle WXZ$ is a 30° - 60° - 90° triangle whose sides are in the ratio $1 : \sqrt{3} : 2$.

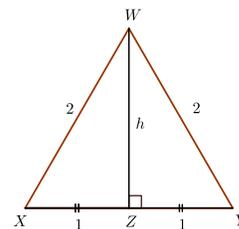
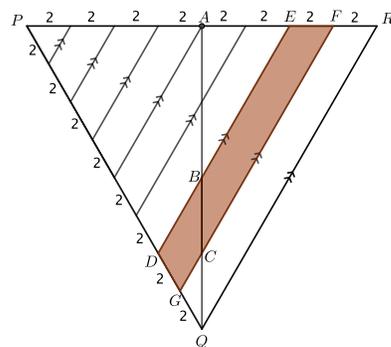
Now back to given $\triangle PQR$, we know that $\angle QPR = \angle PRQ = \angle RQP = 60^\circ$.

In $\triangle DPE$, $\angle DPE = \angle QPR = 60^\circ$ since they are the same angle. Since $DE \parallel RQ$, $\angle PED = \angle PRQ = 60^\circ$ and $\angle EDP = \angle RQP = 60^\circ$. Since all of the angles in $\triangle DPE$ are 60° , it is an equilateral triangle with sides of length 12 cm. Using our above result, the height of $\triangle DPE = \frac{\sqrt{3}}{2} \times 12 = 6\sqrt{3}$. The area of $\triangle DPE = \frac{12 \times 6\sqrt{3}}{2} = 36\sqrt{3}$ cm².

In a similar way, we can show that $\triangle GPF$ is equilateral with side length 14 cm. The height of $\triangle GPF = \frac{\sqrt{3}}{2} \times 14 = 7\sqrt{3}$ and the area of $\triangle GPF = \frac{14 \times 7\sqrt{3}}{2} = 49\sqrt{3}$ cm².

Therefore,

$$\begin{aligned} \text{area of } DEFG &= \text{area of } \triangle GPF - \text{area of } \triangle DPE \\ &= 49\sqrt{3} - 36\sqrt{3} \\ &= 13\sqrt{3} \text{ cm}^2 \end{aligned}$$





Next, we find the area of $CBEF$ by finding the area of $\triangle BEA$ and subtracting it from the area of $\triangle CFA$.

In $\triangle BEA$, $\angle BAE = \angle QAR = 90^\circ$, since the altitude QA is perpendicular to base PR and the two angles represent the same angle. Since $DE \parallel QR$, $\angle BEA = \angle QRP = 60^\circ$. It follows that $\angle EBA = 30^\circ$. Therefore, $\triangle BEA$ is a $30^\circ - 60^\circ - 90^\circ$ triangle and $EA : BA : BE = 1 : \sqrt{3} : 2$. Since $EA = 4$, it follows that $BA = 4\sqrt{3}$ and $BE = 8$. The area of $\triangle BEA = \frac{4 \times 4\sqrt{3}}{2} = 8\sqrt{3} \text{ cm}^2$.

In a similar way, we can show that $\triangle CFA$ is a $30^\circ - 60^\circ - 90^\circ$ triangle with $FA : CA : CF = 1 : \sqrt{3} : 2$. Since $FA = 6$, it follows that $CA = 6\sqrt{3}$ and $CF = 12$. The area of $\triangle CFA = \frac{6 \times 6\sqrt{3}}{2} = 18\sqrt{3} \text{ cm}^2$.

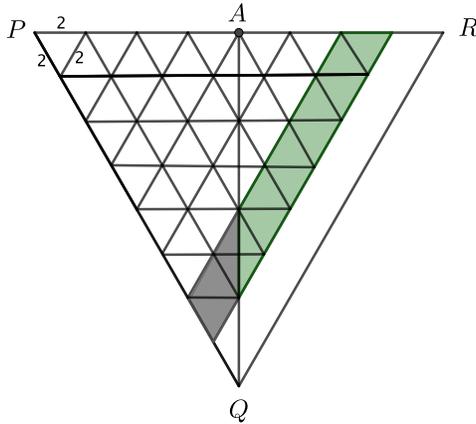
Now, $\text{area of } CBEF = \text{area of } \triangle CFA - \text{area of } \triangle BEA = 18\sqrt{3} - 8\sqrt{3} = 10\sqrt{3} \text{ cm}^2$
 $\text{area of } DBCG = \text{area of } DEFG - \text{area of } CBEF = 13\sqrt{3} - 10\sqrt{3} = 3\sqrt{3} \text{ cm}^2$

Therefore, the ratio of the area of $DBCG$ to the area of $CBEF$ is $3\sqrt{3} : 10\sqrt{3}$ or $3 : 10$.

Solution 2:

In the diagram, we see that we can tile $\triangle QPR$ with the top left equilateral triangle of side length 2 cm. (You may wish to justify that this is indeed an equilateral triangle.)

Then, three equilateral triangles fit in the first trapezoid from the left, 5 equilateral triangles fit in the second trapezoid from the left, 7 equilateral triangles fit in the third trapezoid from the left, 9 equilateral triangles fit in the fourth trapezoid from the left, 11 equilateral triangles fit in the fifth trapezoid from the left, and 13 equilateral triangles fit in the sixth trapezoid from the left.



The shaded trapezoid is the sixth trapezoid from the left and so contains a total of 13 congruent equilateral triangles. To the left of the altitude there are $2 + \frac{1}{2} + \frac{1}{2} = 3$ of the triangles and to the right of the altitude there are $13 - 3 = 10$ of the triangles. Each of the 13 triangles in the shaded trapezoid have the same area.

The ratio of the shaded area to the left of the altitude to the shaded area to the right of the altitude is $3 : 10$.

There are many things in this second solution that the solver may wish to justify.



Problem of the Week

Problem E

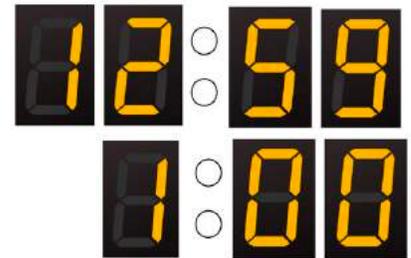
Around the Clock

The digits are displayed on a clock in the following way. Each of the four digit spots consists of seven LEDs (Light-Emitting Diodes) which are turned off or on depending on the digit to be displayed.

When the time changes from “12:00” to “12:01”, the first three digits of the time remain in their same state but four of the LEDs in the final digit change from on to off.



Many more things happen when the clock changes from “12:59” to “1:00”. In the first digit spot, the two LEDs used to display the 1 turn off. In the second digit spot, four of the LEDs that are on to make the 2 turn off and another LED turns on. In the third digit spot, to change from a 5 to a 0, two LEDs need to turn on and one LED needs to turn off. In the fourth digit spot, to change from a 9 to a 0, one LED must turn off and one LED must turn on. In total, to change from “12:59” to “1:00”, a total of $2 + 5 + 3 + 2 = 12$ changes occur.



If the initial time displayed on the clock is “12:00”, how many changes have occurred once the clock displays “12:00” again?

Each of the digits is shown below for your reference.





Problem of the Week

Problem E and Solution

Around the Clock

Problem

The digits are displayed on a clock in the following way. Each of the four digit spots consists of seven LEDs (Light-Emitting Diodes) which are turned off or on depending on the digit to be displayed. When the time changes from “12:00” to “12:01”, the first three digits of the time remain in their same state but four of the LEDs in the final digit change from on to off. Many more things happen when the clock changes from “12:59” to “1:00”. In the first digit spot, the two LEDs used to display the 1 turn off. In the second digit spot, four of the LEDs that are on to make the 2 turn off and another LED turns on. In the third digit spot, to change from a 5 to a 0, two LEDs need to turn on and one LED needs to turn off. In the fourth digit spot, to change from a 9 to a 0, one LED must turn off and one LED must turn on. In total, to change from “12:59” to “1:00”, a total of $2 + 5 + 3 + 2 = 12$ changes occur. If the initial time displayed on the clock is “12:00”, how many changes have occurred once the clock displays “12:00” again?

Solution



This problem has many moving parts. To start, let’s examine all of the possible changes that can take place in any of the four digit spots. The changes are summarized in the chart below.

Example Time	In Changing	# of LEDs changing from OFF to ON	# of LEDs changing from ON to OFF	# of LEDs remaining in the same state	Total # of changes in state
9:59 to 10:00	blank to 1	2	0	5	2
2:50 to 2:51	0 to 1	0	4	3	4
12:59 to 1:00	1 to blank	0	2	5	2
1:59 to 2:00	1 to 2	4	1	2	5
12:59 to 1:00	2 to 1	1	4	2	5
1:52 to 1:53	2 to 3	1	1	5	2
1:53 to 1:54	3 to 4	1	2	4	3
1:54 to 1:55	4 to 5	2	1	4	3
3:59 to 4:00	5 to 0	2	1	4	3
12:45 to 12:46	5 to 6	1	0	6	1
12:46 to 12:47	6 to 7	1	4	2	5
12:47 to 12:48	7 to 8	4	0	3	4
12:48 to 12:49	8 to 9	0	1	6	1
3:49 to 3:50	9 to 0	1	1	5	2



1. Consider the changes in the fourth digit from the left that take place in 12 hours.

Every 10 minutes the fourth digit changes from

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 0$$

By referring to the table, there are a total of $4 + 5 + 2 + 3 + 3 + 1 + 5 + 4 + 1 + 2$ or 30 changes in the fourth digit every 10 minutes. Every hour, the 30 changes happen 6 times or a total of $6 \times 30 = 180$ times. In 12 hours, the 180 changes take place 12 times for a total of 2160 changes in the fourth digit.

2. Consider the changes in the third digit from the left that take place in 12 hours.

Every hour the third digit changes from

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 0$$

By referring to the table, there are a total of $4 + 5 + 2 + 3 + 3 + 3$ or 20 changes in the third digit every hour. Since we are considering 12 hours, there are a total of 12×20 or 240 changes in the third digit.

3. Consider the changes in the second digit from the left that take place in 12 hours.

The second digit changes from

$$2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 0 \rightarrow 1 \rightarrow 2$$

By referring to the table, there are a total of $5 + 5 + 2 + 3 + 3 + 1 + 5 + 4 + 1 + 2 + 4 + 5$ or 40 changes in the second digit every 12 hours.

4. Consider the changes in the leftmost digit that take place in 12 hours.

The leftmost digit changes from

$$1 \rightarrow \text{blank} \rightarrow 1$$

By referring to the table, there are a total of $2 + 2$ or 4 changes in the leftmost digit every 12 hours.

Combining the results from the four cases, there are a total of $2160 + 240 + 40 + 4$ or 2444 changes in the state of the LEDs in the 12 hour period.



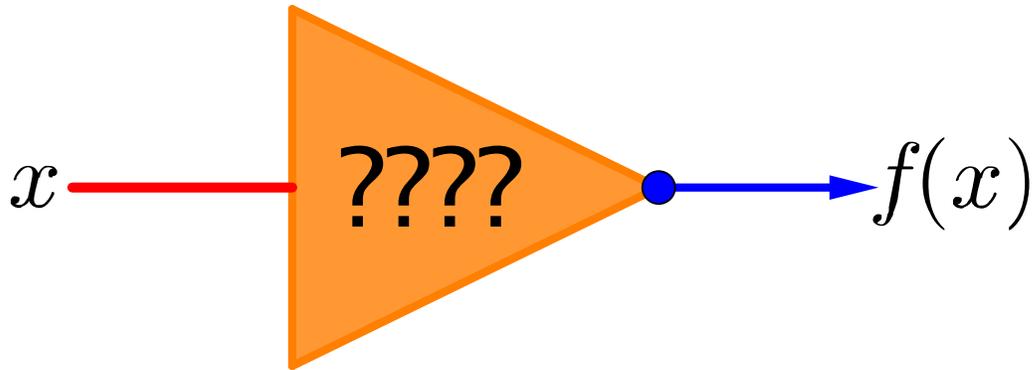
Problem of the Week

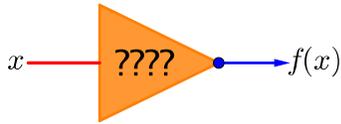
Problem E

Functionally Possible

The function $f(x) = x^5 - 3x^4 + ax^3 - x^2 + bx - 2$ has a value of 5 when $x = 3$.

Determine the value of the function when $x = -3$.





Problem of the Week

Problem E and Solution

Functionally Possible

Problem

The function $f(x) = x^5 - 3x^4 + ax^3 - x^2 + bx - 2$ has a value of 5 when $x = 3$.

Determine the value of the function when $x = -3$.

Solution

We know that the function has a value of 5 when $x = 3$. Therefore, $f(3) = 5$.

$$\begin{aligned} f(3) &= 5 \\ (3)^5 - 3(3)^4 + a(3)^3 - (3)^2 + b(3) - 2 &= 5 \\ 243 - 243 + 27a - 9 + 3b - 2 &= 5 \\ 27a + 3b &= 16 \end{aligned} \quad (1)$$

At this point we seem to have used up the given information. Maybe we can learn more by looking at precisely what we are asked to determine.

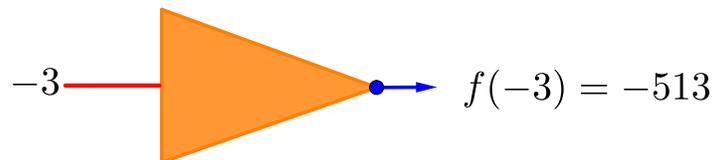
In this problem, we want the value of the function when $x = -3$. In other words, we want $f(-3)$.

$$\begin{aligned} f(-3) &= (-3)^5 - 3(-3)^4 + a(-3)^3 - (-3)^2 + b(-3) - 2 \\ &= -243 - 243 - 27a - 9 - 3b - 2 \\ &= -27a - 3b - 497 \end{aligned}$$

But from (1) above, $27a + 3b = 16$ so

$$f(-3) = -27a - 3b - 497 = -(27a + 3b) - 497 = -16 - 497 = -513.$$

Therefore, the value of the function is -513 when $x = -3$.



We are not given enough information to find the precise values of a and b but enough information is given to solve the problem.



Problem of the Week

Problem E

CUBES

Certain numbers have interesting properties. For example, $1^3 + 5^3 + 3^3 = 153$. That is, the sum of the cubes of the individual digits of the positive integer 153 is the number itself. This may lead you to ask a question like, “Are there other such numbers?” (Yes there are, but that is not our concern today.)

The number 512 stands alone as a three-digit positive integer with three different digits such that the cube of the sum of the digits equals the number itself. That is, $(5 + 1 + 2)^3 = 512$. This is the only three-digit positive integer with three distinct digits that has this property.

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number.

That is, find all five-digit positive integers of the form *CUBES* with distinct digits such that

$$(C + U + B + E + S)^3 = CUBES$$



Problem of the Week

$(C+U+B+E+S)^3 = CUBES$ Problem E and Solution

CUBES

Problem

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number. That is, find all five-digit positive integers of the form $CUBES$ with distinct digits such that $(C + U + B + E + S)^3 = CUBES$.

Solution

A straightforward approach to solving this problem is to determine the smallest possible number and the largest possible number. Then, work at finding the numbers in that range that satisfies the given property.

The smallest five-digit number with distinct digits is 10 234. Since $\sqrt[3]{10\,234} \doteq 21.7$, the smallest number to consider is $22^3 = 10\,648$. The largest sum of five distinct digits is $9 + 8 + 7 + 6 + 5 = 35$, so the largest number to consider is $35^3 = 42\,875$. The possibilities, if any exist, are from 22^3 to 35^3 . We need to examine these cubes to find the solution.

Number	Number ³	Sum of the Digits	Has the Property?
22	10 648	19	no, $22 \neq 19$
23	12 167		no, digits not distinct
24	13 824	18	no, $24 \neq 18$
25	15 625		no, digits not distinct
26	17 576		no, digits not distinct
27	19 683	27	Yes, $(1 + 9 + 6 + 8 + 3)^3 = 19683$
28	21 952		no, digits not distinct
29	24 389	26	no, $29 \neq 26$
30	27 000		no, digits not distinct
31	29 791		no, digits not distinct
32	32 768	26	no, $32 \neq 26$
33	35 937		no, digits not distinct
34	39 304		no, digits not distinct
35	42 875	26	no, $35 \neq 26$

Since we have examined all possibilities, we can conclude that 19 683 is the only five-digit positive integer with distinct digits such that the cube of the sum of the digits of the number equals the original number.



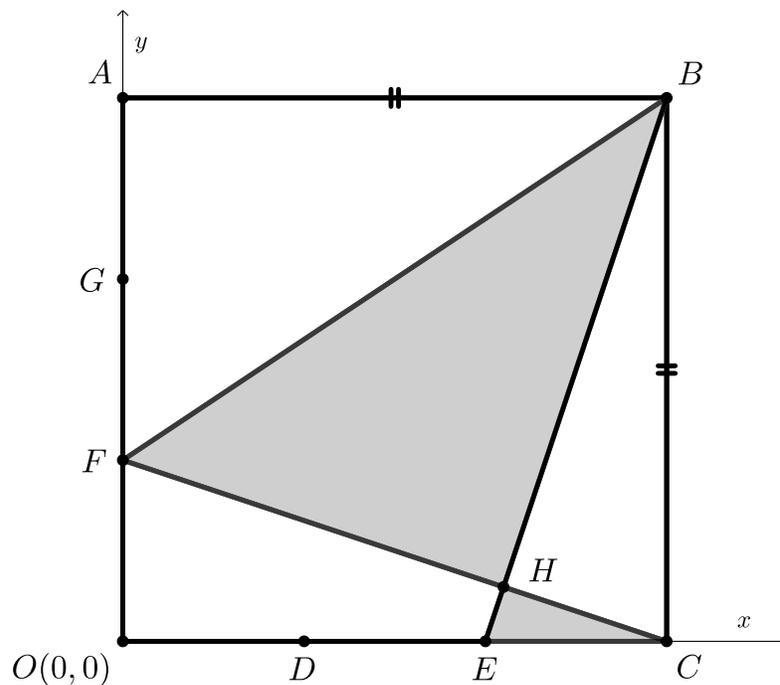
Problem of the Week

Problem E

Maybe One-Third?

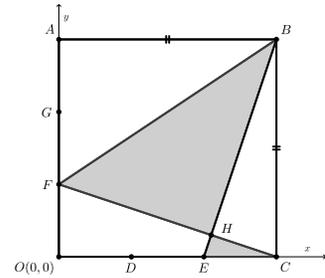
In the diagram, square $OABC$ is positioned with O at the origin $(0,0)$, A on the positive y -axis, C on the positive x -axis, and B in the first quadrant. Side OA is trisected by points F and G so that $OF = FG = GA = 100$. Side OC is trisected by points D and E so that $OD = DE = EC = 100$. Line segment BE intersects line segment CF at H .

If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?





Problem of the Week Problem E and Solution Maybe One-Third?



Problem

In the diagram, square $OABC$ is positioned with O at the origin $(0, 0)$, A on the positive y -axis, C on the positive x -axis, and B in the first quadrant. Side OA is trisected by points F and G so that $OF = FG = GA = 100$. Side OC is trisected by points D and E so that $OD = DE = EC = 100$. Line segment BE intersects line segment CF at H . If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?

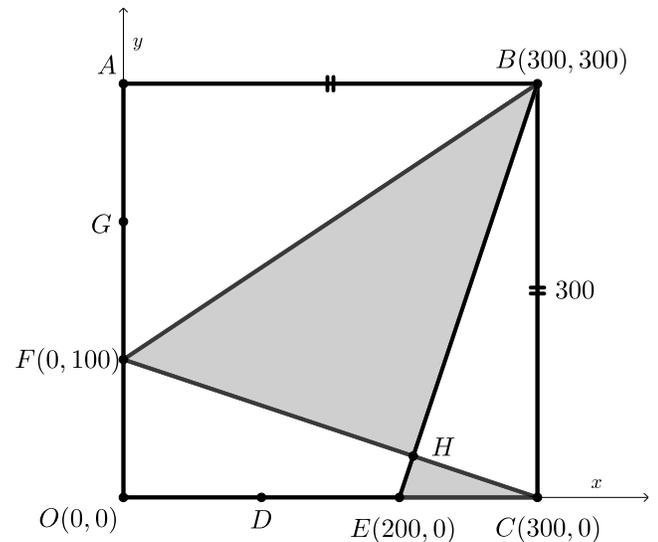
Solution

Since $OF = 100$ and F is on the positive y -axis, the coordinates of F are $(0, 100)$.

Since $OD = DE = 100$, it follows that $OE = 200$. Since E is on the positive x -axis, the coordinates of E are $(200, 0)$.

Since $OD = DE = EC = 100$, it follows that the side length of the square is $OC = 300$. Since C is on the positive x -axis, the coordinates of C are $(300, 0)$.

It then follows that the coordinates of B are $(300, 300)$.



The diagram has been updated to reflect the new information.

We will proceed to find the coordinates of H .

Find the equation of the line through $B(300, 300)$ and $E(200, 0)$.

The slope of $BE = \frac{300-0}{300-200} = 3$. We substitute $x = 200$, $y = 0$ and $m = 3$ into $y = mx + b$ to find b . Then $0 = 3(200) + b$ and $b = -600$ follows. The equation of the line through BE is $y = 3x - 600$. (1)

Find the equation of the line through $C(300, 0)$ and $F(0, 100)$.

The slope of $CF = \frac{100-0}{0-300} = -\frac{1}{3}$. Since $F(0, 100)$ is on the y -axis, the y -intercept is 100. It follows that the equation of the line through C and F is $y = -\frac{1}{3}x + 100$. (2)



Find the coordinates of H , the intersection of the two lines.

At the intersection, the x -coordinates are equal and the y -coordinates are equal. In (1) and (2), since $y = y$, then

$$3x - 600 = -\frac{1}{3}x + 100 \Rightarrow 9x - 1800 = -x + 300 \Rightarrow 10x = 2100 \Rightarrow x = 210$$

Substituting $x = 210$ into (1), $y = 3(210) - 600 = 30$. Therefore, the coordinates of H are $(210, 30)$.

At this point we could follow one of two approaches. The first approach would be to find the area of the shaded regions indirectly, by first determining the area of the unshaded regions and then subtracting this from the area of the square. We will leave this approach to the solver.

Our second approach, which is below, is to calculate the areas of the shaded triangles directly.

The slope of BE is 3 and the slope of CF is $-\frac{1}{3}$. Since these slopes are negative reciprocals, we know that $BE \perp CF$. It follows that $\triangle BHF$ and $\triangle CHE$ are right-angled triangles.

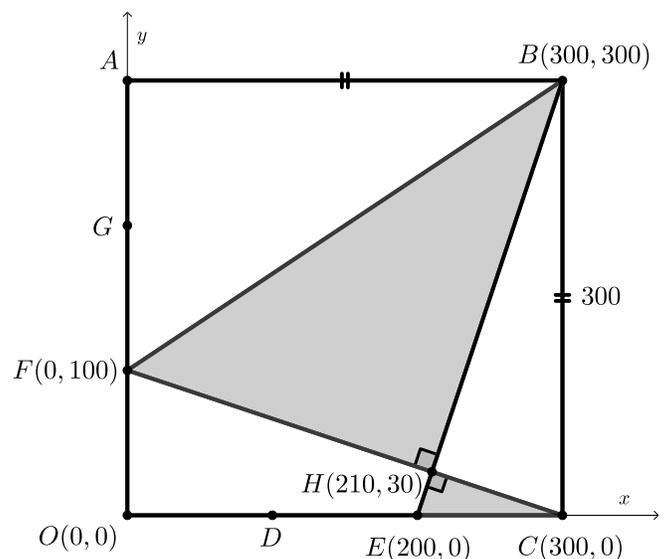
We will now proceed with finding the side lengths necessary to calculate the area of each shaded triangle.

We first find the area of $\triangle BHF$.

$$\begin{aligned} BH &= \sqrt{(300 - 210)^2 + (300 - 30)^2} \\ &= \sqrt{90^2 + 270^2} \\ &= \sqrt{90^2(1 + 3^2)} \\ &= 90\sqrt{10} \end{aligned}$$

$$\begin{aligned} HF &= \sqrt{(0 - 210)^2 + (100 - 30)^2} \\ &= \sqrt{210^2 + 70^2} \\ &= \sqrt{70^2(3^2 + 1)} \\ &= 70\sqrt{10} \end{aligned}$$

$$\begin{aligned} \text{Area } \triangle BHF &= BH \times HF \div 2 \\ &= 90\sqrt{10} \times 70\sqrt{10} \div 2 \\ &= 31\,500 \end{aligned}$$



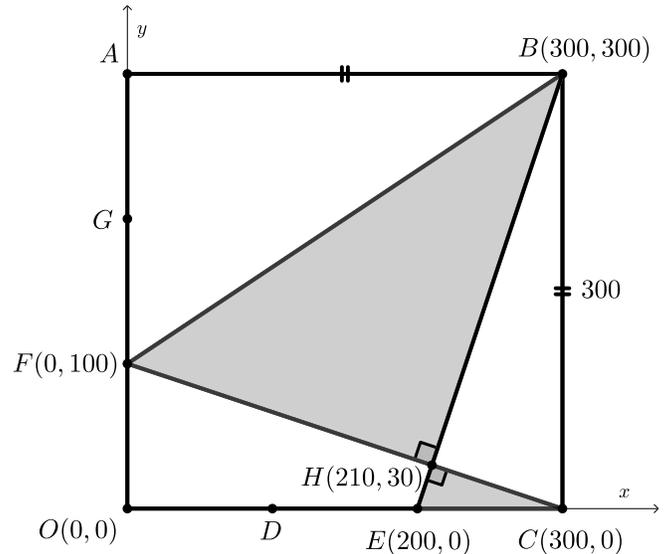


Next we find the area of $\triangle CHE$.

$$\begin{aligned} CH &= \sqrt{(300 - 210)^2 + (0 - 30)^2} \\ &= \sqrt{90^2 + 30^2} \\ &= \sqrt{30^2(3^2 + 1)} \\ &= 30\sqrt{10} \end{aligned}$$

$$\begin{aligned} HE &= \sqrt{(210 - 200)^2 + (30 - 0)^2} \\ &= \sqrt{10^2 + 30^2} \\ &= \sqrt{10^2(1 + 3^2)} \\ &= 10\sqrt{10} \end{aligned}$$

$$\begin{aligned} \text{Area } \triangle CHE &= CH \times HE \div 2 \\ &= 30\sqrt{10} \times 10\sqrt{10} \div 2 \\ &= 1500 \end{aligned}$$



We can now calculate the total area shaded, the area of square $OABC$, and the fraction of the area of the square that is shaded.

$$\begin{aligned} \text{Total Area Shaded} &= \text{Area } \triangle BHF + \text{Area } \triangle CHE \\ &= 31\,500 + 1500 \\ &= 33\,000 \end{aligned}$$

$$\begin{aligned} \text{Area } OABC &= OA \times OC \\ &= 300 \times 300 \\ &= 90\,000 \end{aligned}$$

$$\begin{aligned} \text{Fraction of Total Area Shaded} &= \frac{\text{Area } \triangle BHF}{\text{Area } OABC} \\ &= \frac{33\,000}{90\,000} \\ &= \frac{11}{30} \end{aligned}$$

Therefore, $\frac{11}{30}$ of the total area of the square is shaded. This, in fact, is more than one-third.

Data Management (D)



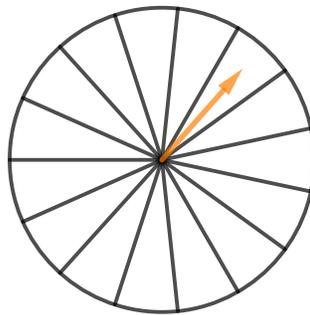


Problem of the Week

Problem E

Spin, Spin, Spin

A spinner is divided into 15 equal sections. Each section is coloured either red, green, or yellow. An arrow is attached to the centre of the spinner. Jordan spins the arrow 3 times. If there is a 48.8% chance of landing on red in at least one of the three spins, how many red sections are there?



You may use a known result from probability theory. If the probability of event A occurring is a , the probability of event B occurring is b , the probability of event C occurring is c , and the results are not dependent on each other, then the probability of all three events happening is $a \times b \times c$.





Problem of the Week

Problem E and Solution

Spin, Spin, Spin

Problem

A spinner is divided into 15 equal sections. Each section is coloured either red, green, or yellow. An arrow is attached to the centre of the spinner. Jordan spins the arrow 3 times. If there is a 48.8% chance of landing on red in at least one of the three spins, how many red sections are there?

Solution

There will be two solutions presented. In the second solution, we will need to make use of the Factor Theorem, which is typically taught in a Grade 12 math course. In both we will use the given fact that if the probability of event A occurring is a , the probability of event B occurring is b , the probability of event C occurring is c , and the results are not dependent on each other, then the probability of all three events happening is $a \times b \times c$.

Solution 1:

Let n be the number of non-red sections. Therefore, for each spin, the probability the spinner does not land on a red section is $\frac{n}{15}$. Since the result of each spin does not depend on the previous spin,

$$\begin{aligned} P(\text{no reds in all 3 spins}) &= P(\text{1st not red}) \times P(\text{2nd not red}) \times P(\text{3rd not red}) \\ &= \frac{n}{15} \times \frac{n}{15} \times \frac{n}{15} \\ &= \left(\frac{n}{15}\right)^3 \end{aligned}$$

The probability of landing on at least one red in three spins is 0.488. So, the probability of not landing on red in any of the three spins is $1 - 0.488 = 0.512$.

That is,

$$\begin{aligned} \left(\frac{n}{15}\right)^3 &= 0.512 \\ \left(\frac{n}{15}\right)^3 &= \frac{512}{1000} \\ \frac{n}{15} &= \frac{8}{10} \\ n &= 12 \end{aligned}$$

Since n is the number of non-red sections, there are $15 - 12 = 3$ red sections.

**Solution 2:**

Let r be the number of red sections. Therefore, the number of non-red sections is $15 - r$. Also, for each spin, the probability the spinner lands on a red section is $\frac{r}{15}$ and the probability the spinner does not land on a red section is $\frac{15-r}{15}$.

If Jordan lands on red in at least one spin, then the first red occurs on his first spin, second spin, or third spin.

If Jordan lands on the first red on his first spin, then on his first spin he spins a red, on his second spin he spins any colour, and on his third spin he spins any colour. Since the results of each spin do not depend on each other, the probability that the first red occurs on his first spin is $P(1) = \frac{r}{15} \times \frac{15}{15} \times \frac{15}{15} = \frac{r}{15}$.

If Jordan lands on the first red on the second spin, then on his first spin he did not spin a red, on his second spin he spins a red, and on his third spin he spins any colour. The probability of this is $P(2) = \frac{15-r}{15} \times \frac{r}{15} \times \frac{15}{15} = \frac{(15-r)r}{15^2}$.

If Jordan lands on the first red on the third spin, then on his first two spins he did not spin a red and on this third spin he spins a red. The probability of this is

$$P(3) = \frac{15-r}{15} \times \frac{15-r}{15} \times \frac{r}{15} = \frac{(15-r)^2 r}{15^3}.$$

The probability of getting at least one red in the three spins is equal to $P(1) + P(2) + P(3)$.

That is,

$$\begin{aligned} \frac{r}{15} + \frac{(15-r)r}{15^2} + \frac{(15-r)^2 r}{15^3} &= 0.488 \\ 225r + 15(15-r)r + (15-r)^2 r &= 1647 \\ r^3 - 45r^2 + 675r - 1647 &= 0 \\ (r-3)(r^2 - 42r + 549) &= 0 \quad (*\text{see below to see the factoring}) \end{aligned}$$

Therefore, $r - 3 = 0$ or $r^2 - 42r + 549 = 0$. Using the quadratic formula, we see that $r^2 - 42r + 549 = 0$ has no real solution. Therefore, the only solution is $r = 3$.

That is, there are 3 red sections.

***FACTORING THE CUBIC:**

It is easy to verify the factoring is correct by expanding. To determine the factorization, we used the Factor Theorem. This topic would typically be covered in a Grade 12 Math course.

Let $f(r) = r^3 - 45r^2 + 675r - 1647$. Notice that $f(3) = 0$. Therefore, the Factor Theorem tells us that $(r - 3)$ is a factor of $f(r)$. Therefore, $f(r) = (r - 3)(ar^2 + br + c)$, for some real numbers a, b, c . Expanding this, we get

$$f(r) = (r - 3)(ar^2 + br + c) = ar^3 + br^2 + cr - 3ar^2 - 3br - 3c = ar^3 + (b - 3a)r^2 + (c - 3b)r - 3c.$$

Setting the two expressions for $P(r)$ equal, we get

$$r^3 - 45r^2 + 675r - 1647 = ar^3 + (b - 3a)r^2 + (c - 3b)r - 3c.$$

Setting the coefficients for r^3 equal, we get $a = 1$.

Setting the constants equal, we get $-3c = -1647$ or $c = 549$.

Setting the coefficients for r^2 equal, we get $b - 3a = -45$ or $b = -42$.

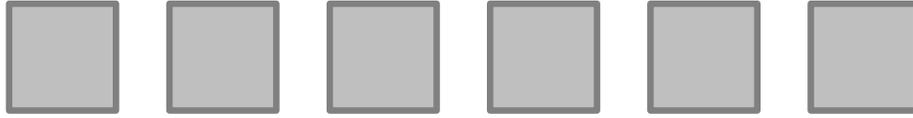
Therefore, $f(r) = (r - 3)(r^2 - 42r + 549)$.



Problem of the Week

Problem E

Access Denied



A six-digit code is required to gain access to a secure building.

You know the following about the correct code:

- only the digits 1, 2, 3, 4, 5, and 6 are used in the correct code and each of these digits appears exactly once;
- the fifth digit in the code is not a 5;
- the sixth digit in the code is not a 6; and
- there is only one correct code.

What is the probability that you are able to enter the correct code?

Note: Often in counting problems, products such as $4 \times 3 \times 2 \times 1$ are encountered. This product can be written using factorial notation, as $4!$. In general, if n is a positive integer, then $n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1$.





Problem of the Week

Problem E and Solution

Access Denied

Problem

A six-digit code is required to gain access to a secure building. You know the following about the correct code: only the digits 1, 2, 3, 4, 5, and 6 are used in the correct code and each of these digits appears exactly once, the fifth digit in the code is not a 5, the sixth digit in the code is not a 6, and there is only one correct code. What is the probability that you are able to enter the correct code?

Solution

Solution 1

In this solution we will determine the number of possibilities directly, examining the possible cases.

1. Neither the 5 nor the 6 appear in the fifth or sixth positions.

There are four possible positions for the 5 and for each of these choices there are 3 possible positions for the 6. There are $3 \times 4 = 12$ ways to place the 5 and 6. Once placed, there are four positions left to fill with the four remaining digits. These can be placed in $4 \times 3 \times 2 \times 1$ or $4!$ ways. Therefore, there are $12 \times 4! = 12 \times 24 = 288$ possible numbers in which neither the 5 or the 6 are in the fifth or sixth positions.

2. The 6 appears in the fifth position and the 5 appears in sixth position.

There is 1 way to place the 5 and 6. The remaining positions can be filled in $4!$ or 24 ways.

3. The 6 is in the fifth position but the 5 is not in the sixth position.

There are 4 ways to place the 5. For each of these, there are $4!$ ways to place the remaining four digits. Therefore, there are $4 \times 4! = 96$ possible numbers in which 6 is in the fifth position, and 5 is not in the sixth position.

4. The 5 is in the sixth position but the 6 is not in fifth position.

There are 4 ways to place the 6. For each of these, there are $4!$ ways to place the remaining four digits. Therefore, there are $4 \times 4! = 96$ possible numbers in which 5 is in the sixth position, and 6 is not in the fifth position.

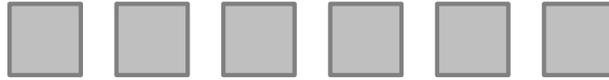
Therefore, there are $288 + 24 + 96 + 96 = 504$ different possible codes that satisfy the given conditions. The probability of entering the correct code is $\frac{1}{504}$.



Solution 2

In this solution we will determine the number of possibilities indirectly.

We will calculate the total possible number of codes using each of the digits 1, 2, 3, 4, 5, and 6 exactly once without considering the other restrictions. We will then subtract the codes which are not permitted.



The total number of different six-digit codes using the numbers 1, 2, 3, 4, 5, and 6 exactly once is $6!$ or 720. But this includes the following possibilities which are **not** permitted.

1. The 5 is in the fifth position and the 6 is in the sixth position.



The four remaining digits can be filled in $4!$ or 24 ways. There are 24 codes in which the 5 is in the fifth position and the 6 is in the sixth position.

2. The 5 is in the fifth position but the 6 is not in the sixth position.



Since the 6 cannot be in the sixth position, there are only four positions for the 6. For each of these 4 possible placements of the 6 there are $4!$ ways to place the remaining digits. There are $4 \times 4! = 4 \times 24 = 96$ codes in which the 5 is in the fifth position but the 6 is not in the sixth position.

3. The 6 is in the sixth position but the 5 is not in the fifth position.



Since the 5 cannot be in the fifth position, there are only four positions for the 5. For each of these 4 possible placements of the 5 there are $4!$ ways to place the remaining digits. There are $4 \times 4! = 4 \times 24 = 96$ codes in which the 6 is in the sixth position but the 5 is not in the fifth position.

Therefore, there are $720 - 24 - 96 - 96 = 504$ different possible codes that satisfy the given conditions. The probability of entering the correct code is $\frac{1}{504}$.

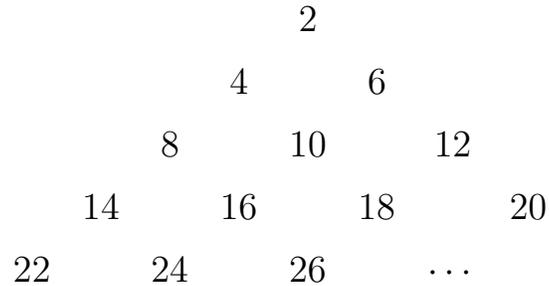


Problem of the Week

Problem E

This is the Year 2

The positive even integers are arranged in increasing order in a triangle, as shown:



Each row contains one more number than the previous row. One of the rows of the triangle contains the number 2020. Determine the sum of the numbers in the row that contains 2020.



Did you know?

There is a quick way to calculate the sum $1 + 2 + 3 + 4 + \dots + 19 + 20$?

$$1 + 2 + 3 + 4 + \dots + 19 + 20 = \frac{(20)(20 + 1)}{2} = 210$$

In general, it can be shown that if n is a positive integer, then the sum of the integers from 1 to n is $S = 1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$.



Problem of the Week

Problem E and Solution

This is the Year 2

Problem

The positive even integers are arranged in increasing order in a triangle, as shown:

			2					
			4		6			
		8		10		12		
	14		16		18		20	
		22		24		26		...

Each row contains one more number than the previous row. One of the rows of the triangle contains the number 2020. Determine the sum of the numbers in the row that contains 2020.

Solution

To begin the solution, we make an observation: the row number in the triangle is the same as the number of numbers in that particular row. For example, row 1 contains 1 number, row 2 contains 2 numbers, and row r contains r numbers. If we add up all of the row numbers from row 1 to row r we get the number of numbers in the table and the largest number in the row is $2r$.

For example, if we calculate the sum $1 + 2 + 3 + 4$, we get 10. This number corresponds to the number of numbers in the first four rows of the triangle and the largest number in the fourth row is $2 \times 10 = 20$.

Let n represent the row number of the row that contains the number 2020. Since $2020 = 2 \times 1010$ we need to find the row where the 1010th number occurs.

Using the formula for the sum of the integers from 1 to n , we want $\frac{n(n+1)}{2} \geq 1010$. Multiplying by 2, the equation simplifies to $n(n+1) \geq 2020$.

We will use trial and error to find the possible value of n . Finding the $\sqrt{2020}$ will get us a good place to start checking values of n . ($\sqrt{2020} \approx 44.9$.)

When $n = 44$, $44 \times 45 = 1980 < 2020$ and $\frac{44 \times 45}{2} = 990$.

When $n = 45$, $45 \times 46 = 2070 > 2020$ and $\frac{45 \times 46}{2} = 1035$.

These guesses are quite useful. When there are 44 rows in the triangle, the last number in the 44th row is $990 \times 2 = 1980$. So the first number in the 45th row is 1982. When there are 45 rows in the triangle, the last number in the 45th row is $2 \times 1035 = 2070$. We want the sum of the numbers in the 45th row. To do this we will add all of the even integers from 2 to 2020. But this sum includes extra even integers from 2 to 1980. We will use our formula to compute this second sum and subtract it from the first sum to obtain the desired result.



$$\begin{aligned}\text{sum} &= 1982 + 1984 + 1986 + \cdots + 2066 + 2068 + 2070 \\ &= (\mathbf{2 + 4 + 6 + \cdots + 1976 + 1978 + 1980} + 1982 + 1984 + 1986 + \cdots + 2066 + 2068 + 2070) \\ &\quad - (\mathbf{2 + 4 + 6 + \cdots + 1976 + 1978 + 1980}) \\ &= (2(\mathbf{1 + 2 + 3 + \cdots + 988 + 989 + 990} + 991 + 992 + 993 + \cdots + 1033 + 1034 + 1035)) \\ &\quad - (2(\mathbf{1 + 2 + 3 + \cdots + 988 + 989 + 990})) \\ &= 2 \times \left(\frac{1035 \times 1036}{2} \right) - 2 \times \left(\frac{990 \times 991}{2} \right) \\ &= 91\,170\end{aligned}$$

The sum of the numbers in the row containing 2020 is 91 170.



Problem of the Week

Problem E

Sum Product Equality

The number 8 is the sum and product of the numbers in the collection of four positive integers $(1, 1, 2, 4)$, since $1 + 1 + 2 + 4 = 8$ and $1 \times 1 \times 2 \times 4 = 8$.

The number 2020 can also be made from a collection of n positive integers, with $n > 1$, that multiply to 2020 and add to 2020.

Determine all possible values for n .

$$+ = \times$$





$$+ = \times$$

Problem of the Week

Problem E and Solution

Sum Product Equality

Problem

The number 8 is the sum and product of the numbers in the collection of four positive integers $(1, 1, 2, 4)$, since $1 + 1 + 2 + 4 = 8$ and $1 \times 1 \times 2 \times 4 = 8$. The number 2020 can be made from a collection of n positive integers, with $n > 1$, that multiply to 2020 and add to 2020. Determine all possible values for n .

Solution

To form such a collection of integers, our strategy is to determine all collections of integers larger than 1 whose product is 2020, and then for each collection add enough 1s to make the sum of the numbers in the collection 2020.

Since we want to consider integers whose product is 2020, we should find the divisors of 2020. The prime factorization of 2020 is $2020 = 2 \times 2 \times 5 \times 101$. We can use this factorization to determine all collections of integers greater than 1 that multiply to 2020. The only collection with 4 numbers is $(2, 2, 5, 101)$. The collections with 3 numbers are $(4, 5, 101)$, $(2, 10, 101)$, $(2, 5, 202)$, and $(2, 2, 505)$. A systematic count shows that the only collections with 2 numbers are $(4, 505)$, $(5, 404)$, $(10, 202)$, $(2, 1010)$, and $(20, 101)$. It is left for the reader to verify this. We will not consider the collection (2020) , as it is given that the number of integers in the collection is to be greater than 1.

We show these collections in the first column of the table below. We also determine the sum of the integers in this collection, the number of 1s needed (subtract the sum from 2020), and the value of n , which is equal to the number of integers in first column plus the entry in third column.

Collection of integers greater than 1	Sum of integers greater than 1 in collection	Number of 1s needed to sum to 2020	Value of n
$(2, 2, 5, 101)$	110	1910	$1910 + 4 = 1914$
$(4, 5, 101)$	110	1910	$1910 + 3 = 1913$
$(2, 10, 101)$	113	1907	$1907 + 3 = 1910$
$(2, 5, 202)$	209	1811	$1811 + 3 = 1814$
$(2, 2, 505)$	509	1511	$1511 + 3 = 1514$
$(4, 505)$	509	1511	$1511 + 2 = 1513$
$(5, 404)$	409	1611	$1611 + 2 = 1613$
$(10, 202)$	212	1808	$1808 + 2 = 1810$
$(2, 1010)$	1012	1008	$1008 + 2 = 1010$
$(20, 101)$	121	1899	$1899 + 2 = 1901$

Therefore, the possible values for n are 1010, 1513, 1514, 1613, 1810, 1814, 1901, 1910, 1913, and 1914.



Problem of the Week

Problem E

What's Your Unlucky Number?

Everyone has a lucky number. Sue Perstitious does not have a lucky number but considers the number 13 to be unlucky.

Three bags each contain tokens. The green bag contains 20 round green tokens, each with a different integer from 1 to 20. The red bag contains 12 triangular red tokens, each with a different integer from 1 to 12. The blue bag contains 8 square blue tokens, each with a different integer from 1 to 8.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $20 \times 12 \times 8 = 1920$ different combinations of tokens that can be created by selecting one token from each bag. Note that the order of selection does not matter. Also note that selecting the 7 red token, the 5 blue token and 1 green token is different than selecting the 5 red token, 7 blue token and the 1 green token.



Sue selects one token from each bag. What is the probability that the sum of the numbers selected is divisible by 13, her unlucky number?



Problem of the Week

Problem E and Solution

What's Your Unlucky Number?

Problem

Everyone has a lucky number. Sue Perstitious does not have a lucky number but considers the number 13 to be unlucky. Three bags each contain tokens. The green bag contains 20 round green tokens, each with a different integer from 1 to 20. The red bag contains 12 triangular red tokens, each with a different integer from 1 to 12. The blue bag contains 8 square blue tokens, each with a different integer from 1 to 8.

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Sue selects one token from each bag. What is the probability that the sum of the numbers selected is divisible by 13, her unlucky number?

Solution

There are 20 different numbers which can be selected from the green bag, 12 different numbers which can be selected from the red bag, and 8 different numbers which can be selected from the blue bag. So there are $20 \times 12 \times 8 = 1920$ different combinations of numbers which can be selected from the three bags.

Let (g, r, b) represent the outcome of a selection where g is the number on the token selected from the green bag, r is the number on the token selected from the red bag and b is the number on the token selected from the blue bag. Also, $1 \leq g \leq 20$, $1 \leq r \leq 12$, and $1 \leq b \leq 8$, for integers g, r, b .

Numbers that are divisible by 13 are 13, 26, 39, 52, \dots . The maximum sum that can be reached any selection is $8 + 12 + 20 = 40$. To count the number of possibilities for sums which are divisible by 13, we will consider three cases: a sum of 13, a sum of 26 and a sum of 39. Within the first two cases, we will look at sub-cases based on the possible outcome for the 8 possible selections from the blue bag.

1. The sum of the numbers on the 3 tokens is 13.

• 1 is on the token selected from the blue bag

The sum of the numbers on the other two tokens is 12. Selecting the 12 token from the red bag is not possible since the number on the token selected from the green bag must be at least 1. The possibilities for (g, r, b) are $(1, 11, 1), (2, 10, 1), (3, 9, 1), \dots, (11, 1, 1)$, 11 possibilities in total.

• 2 is on the token selected from the blue bag

The sum of the numbers on the other two tokens is 11. Selecting the 11 or 12 token from the red bag is not possible since the number on the token selected from the green bag must be at least 1. The possibilities for (g, r, b) are $(1, 10, 2), (2, 9, 2), (3, 8, 2), \dots, (10, 1, 2)$, 10 possibilities in total.



- **3 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 10. Using reasoning similar to the preceding two cases, the possibilities for (g, r, b) are $(1, 9, 3), (2, 8, 3), (3, 7, 3), \dots, (9, 1, 3)$, 9 possibilities in total.

- **4 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 9. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 8, 4), (2, 7, 4), (3, 6, 4), \dots, (8, 1, 4)$, 8 possibilities in total.

- **5 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 8. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 7, 5), (2, 6, 5), (3, 5, 5), \dots, (7, 1, 5)$, 7 possibilities in total.

- **6 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 7. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 6, 6), (2, 5, 6), (3, 4, 6), \dots, (6, 1, 6)$, 6 possibilities in total.

- **7 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 6. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 5, 7), (2, 4, 7), (3, 3, 7), (4, 2, 7), (5, 1, 7)$, 5 possibilities in total.

- **8 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 5. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 4, 8), (2, 3, 8), (3, 2, 8), (4, 1, 8)$, 4 possibilities in total.

Summing the results from the 8 cases, there are $11 + 10 + 9 + \dots + 5 + 4 = 60$ combinations so that the sum of numbers on the 3 tokens is 13.

2. The sum of the numbers on the 3 tokens is 26.

- **1 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 25. The largest number possible from the green bag is 20 so the smallest possible number from the red bag would be 5. The largest number possible from the red bag is 12 so the smallest possible number from the green bag would be 13. The numbers from the green bag go from 20 to 13 while the numbers from the red bag go from 5 to 12. The possibilities for (g, r, b) are $(20, 5, 1), (19, 6, 1), (18, 7, 1), \dots, (13, 12, 1)$, 8 possibilities in total.

- **2 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 24. Using similar reasoning to the first case in this section, the possibilities for (g, r, b) are $(20, 4, 2), (19, 5, 2), (18, 6, 2), \dots, (12, 12, 2)$, 9 possibilities in total.

- **3 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 23. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are $(20, 3, 3), (19, 4, 3), (18, 5, 3), \dots, (11, 12, 3)$, 10 possibilities in total.



- **4 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 22. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are $(20, 2, 4), (19, 3, 4), (18, 4, 4), \dots, (10, 12, 4)$, 11 possibilities in total.

- **5 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 21. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are $(20, 1, 5), (19, 2, 5), (18, 3, 5), \dots, (9, 12, 5)$, 12 possibilities in total.

- **6 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 20. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 8. The highest number possible from the green bag is 19 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are $(19, 1, 6), (18, 2, 6), (17, 3, 6), \dots, (8, 12, 6)$, 12 possibilities in total.

- **7 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 19. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 7. The highest number possible from the green bag is 18 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are $(18, 1, 7), (17, 2, 7), (16, 3, 7), \dots, (7, 12, 7)$, 12 possibilities in total.

- **8 is on the token selected from the blue bag**

The sum of the numbers on the other two tokens is 18. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 6. The highest number possible from the green bag is 17 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are $(17, 1, 8), (16, 2, 8), (15, 3, 8), \dots, (6, 12, 8)$, 12 possibilities in total.

Summing the results from the cases, there are $8 + 9 + 10 + 11 + 4(12) = 86$ combinations so that the sum of the numbers on the 3 tokens is 26.

3. The sum of the numbers on the 3 tokens is 39.

The maximum sum that can be obtained is $8 + 12 + 20 = 40$. A sum of 39 can only be achieved by keeping two of the three tokens at their maximum and reducing the third token to 1 less than its maximum. The possibilities for (g, r, b) are $(20, 12, 7), (20, 11, 8)$ and $(19, 12, 8)$, 3 possibilities in total.

The total number of combinations in which the sum of the numbers on the three tokens is divisible by 13 is $60 + 86 + 3 = 149$.

Therefore, the probability of selecting three tokens with a sum which is divisible by 13 is $\frac{149}{1920}$.

There is less than an 8% chance that the three tokens selected by Sue will sum to a multiple of her unlucky number.



Problem of the Week

Problem E

CUBES

Certain numbers have interesting properties. For example, $1^3 + 5^3 + 3^3 = 153$. That is, the sum of the cubes of the individual digits of the positive integer 153 is the number itself. This may lead you to ask a question like, “Are there other such numbers?” (Yes there are, but that is not our concern today.)

The number 512 stands alone as a three-digit positive integer with three different digits such that the cube of the sum of the digits equals the number itself. That is, $(5 + 1 + 2)^3 = 512$. This is the only three-digit positive integer with three distinct digits that has this property.

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number.

That is, find all five-digit positive integers of the form *CUBES* with distinct digits such that

$$(C + U + B + E + S)^3 = CUBES$$



Problem of the Week

$(C+U+B+E+S)^3 = CUBES$ Problem E and Solution

CUBES

Problem

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number. That is, find all five-digit positive integers of the form $CUBES$ with distinct digits such that $(C + U + B + E + S)^3 = CUBES$.

Solution

A straightforward approach to solving this problem is to determine the smallest possible number and the largest possible number. Then, work at finding the numbers in that range that satisfies the given property.

The smallest five-digit number with distinct digits is 10 234. Since $\sqrt[3]{10\,234} \doteq 21.7$, the smallest number to consider is $22^3 = 10\,648$. The largest sum of five distinct digits is $9 + 8 + 7 + 6 + 5 = 35$, so the largest number to consider is $35^3 = 42\,875$. The possibilities, if any exist, are from 22^3 to 35^3 . We need to examine these cubes to find the solution.

Number	Number ³	Sum of the Digits	Has the Property?
22	10 648	19	no, $22 \neq 19$
23	12 167		no, digits not distinct
24	13 824	18	no, $24 \neq 18$
25	15 625		no, digits not distinct
26	17 576		no, digits not distinct
27	19 683	27	Yes, $(1 + 9 + 6 + 8 + 3)^3 = 19683$
28	21 952		no, digits not distinct
29	24 389	26	no, $29 \neq 26$
30	27 000		no, digits not distinct
31	29 791		no, digits not distinct
32	32 768	26	no, $32 \neq 26$
33	35 937		no, digits not distinct
34	39 304		no, digits not distinct
35	42 875	26	no, $35 \neq 26$

Since we have examined all possibilities, we can conclude that 19 683 is the only five-digit positive integer with distinct digits such that the cube of the sum of the digits of the number equals the original number.

Computational Thinking (C)





Problem of the Week

Problem E

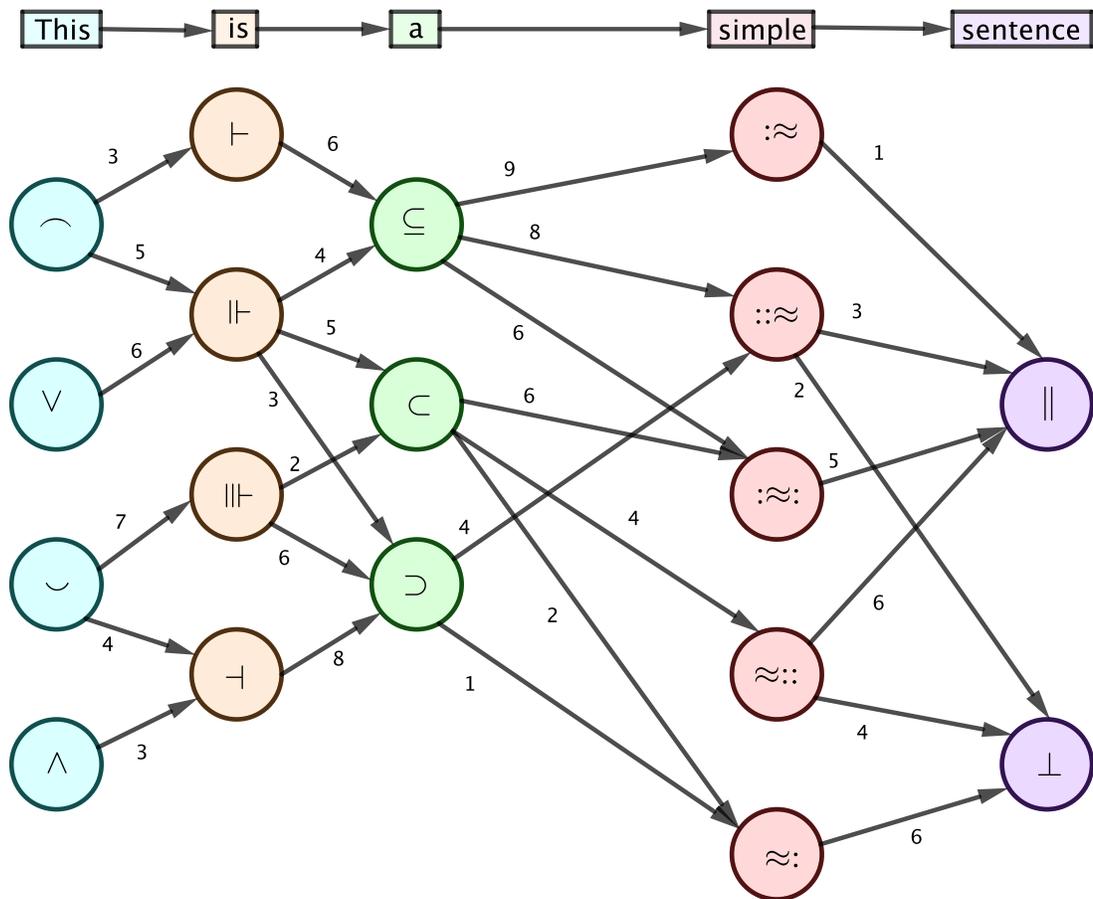
Not So Simple

Chiknskratch is a language that uses symbols rather than words. Elizabeth programs a machine that translates an English sentence to a Chiknskratch sentence one word at a time. However, there are several possible Chiknskratch symbols for each English word! Notice below that the word “This” can be translated using 4 different symbols.

Elizabeth noticed that different symbols occur next to each other at different rates. For example, “smart chicken” is more common than “intelligent chicken”. She gives scores to word pairs: the higher the score, the more common the word pair is.

An English sentence with five words must be translated into five Chiknskratch symbols. In the picture below, arrows labelled with scores connect all valid word pairs for the sentence “This is a simple sentence”. The total score for a translation is the sum of the scores of the four arrows used.

What is the highest possible total score for a translation of this sentence?





Problem of the Week

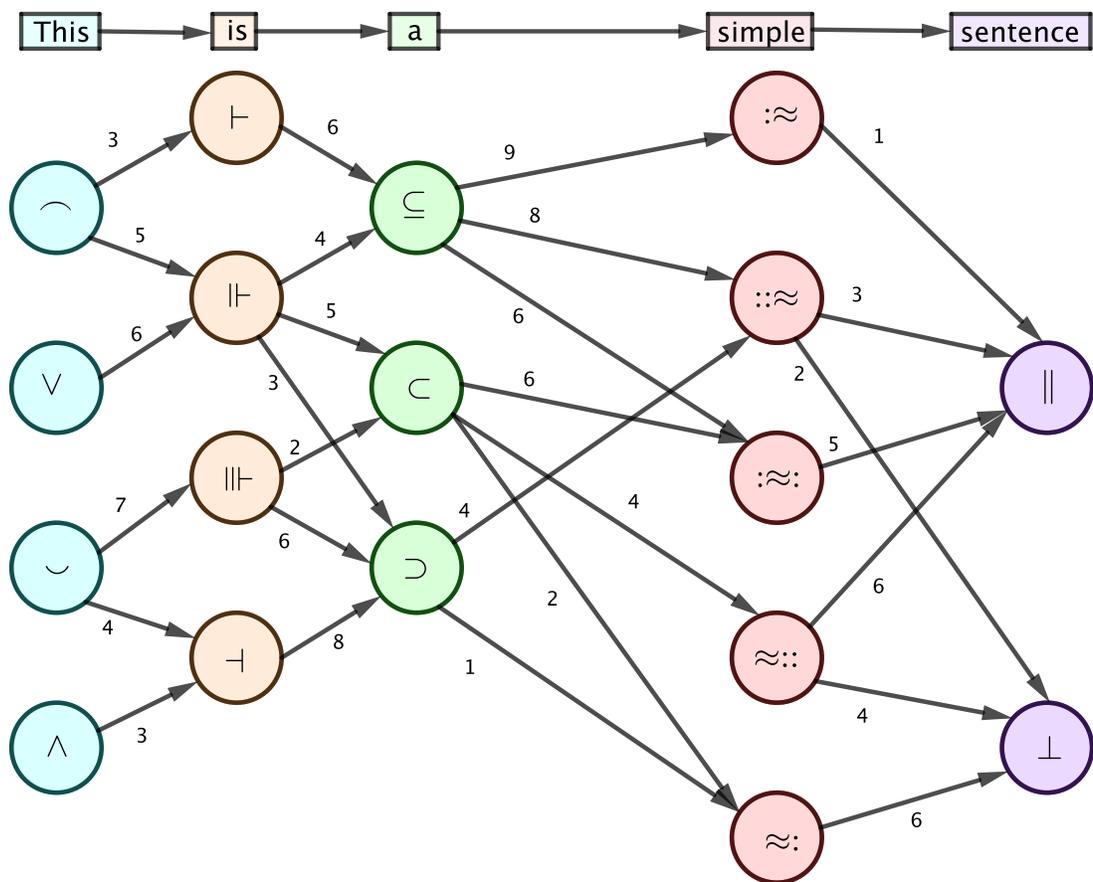
Problem E and Solution

Not So Simple

Problem

(The solution is found on the next page.) Chiknskratch is a language that uses symbols rather than words. Elizabeth programs a machine that translates an English sentence to a Chiknskratch sentence one word at a time. However, there are several possible Chiknskratch symbols for each English word! Notice below that the word “This” can be translated using 4 different symbols. Elizabeth noticed that different symbols occur next to each other at different rates. For example, “smart chicken” is more common than “intelligent chicken”. She gives scores to word pairs: the higher the score, the more common the word pair is. An English sentence with five words must be translated into five Chiknskratch symbols. In the picture below, arrows labelled with scores connect all valid word pairs for the sentence “This is a simple sentence”. The total score for a translation is the sum of the scores of the four arrows used.

What is the highest possible total score for a translation of this sentence?



Solution



The best possible score is 22. This can be achieved by choosing the second, second, second, third, and then finally the first symbol (counting from the top and moving from left-to-right). How can we find this?

We systematically go word by word from left to right assigning a value to each symbol that is the best possible score for a sentence using that symbol.

To begin, we assign a value of 0 to the four possible translations for “This”. Then we consider the four symbols for “is”. We set the value of each corresponding translation of “This is” to be the maximum label of an arrow coming into the corresponding symbol of “is”. Reading from top to bottom, this gives us values of 3, 6, 7, 4 for the values of translations of “This is”.

To assign values to the translations for “This is a”, we consider the three symbols for “a”. For each symbol, we sum of the value on an arrow into the symbol with the corresponding value for “This is” coming from that arrow. We then take the maximum of these sums. Reading from top to bottom, this gives us values of 10, 11 and 13 for translations of “This is a”.

To assign values to the translations for “This is a simple”, we consider the five symbols for “simple”. For each symbol, we sum the value on an arrow into the symbol with the corresponding value for “This is a” coming from that arrow. We then take the maximum of these sums. Reading from top to bottom, this gives us values of 19, 18, 17, 15, and 14 for translations of “This is a simple”.

Finally, to assign values to the translations for “This is a simple sentence”, we consider the two symbols for “sentence”. For each, we sum the value on an arrow into the symbol with the corresponding value for “This is a simple” coming from that arrow. We then take the maximum of these sums. Reading from top to bottom, this gives us values of 22 and 20 for the translations of “This is a simple sentence”.

These “passes” are shown at the end of this solution.

We end with a best possible score for the full sentence is 22. The corresponding translation is “ $\forall \vdash \subset \approx \parallel$ ”. Using this method to find the best score, we only use $7 + 8 + 7 = 22$ additions and some comparisons instead of $(20 + 19) \times 3 = 117$ additions if we checked every possible path through all the nodes. Can you see where these numbers come from?

The Beaver Computing Challenge (BCC):

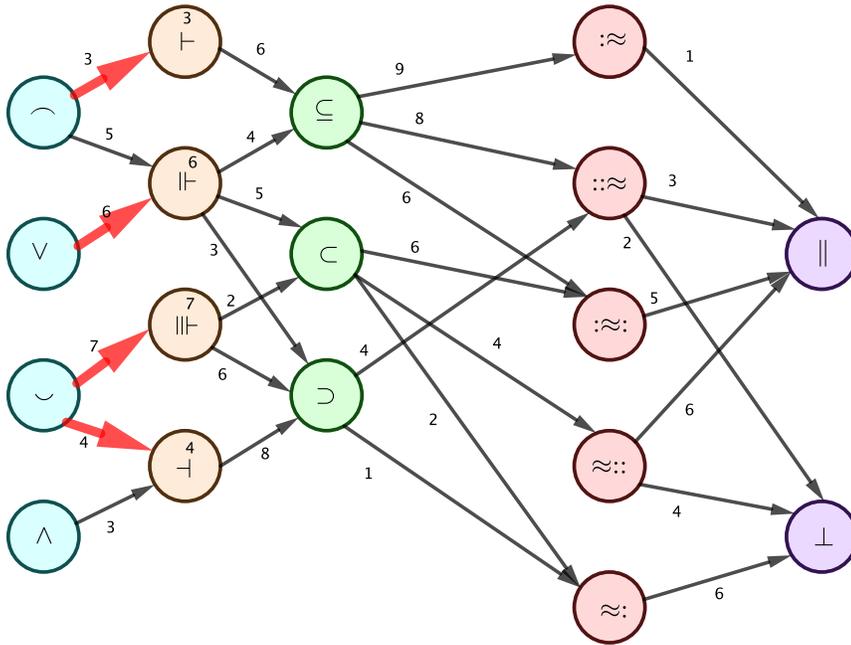
This problem is based on a previous BCC problem. The BCC is designed to get students with little or no previous experience excited about computing. Questions are inspired by topics in computer science and connections to Computer Science are described in the solutions to all past BCC problems. If you enjoyed this problem, you may want to explore the BCC contest further.

Connections to Computer Science:

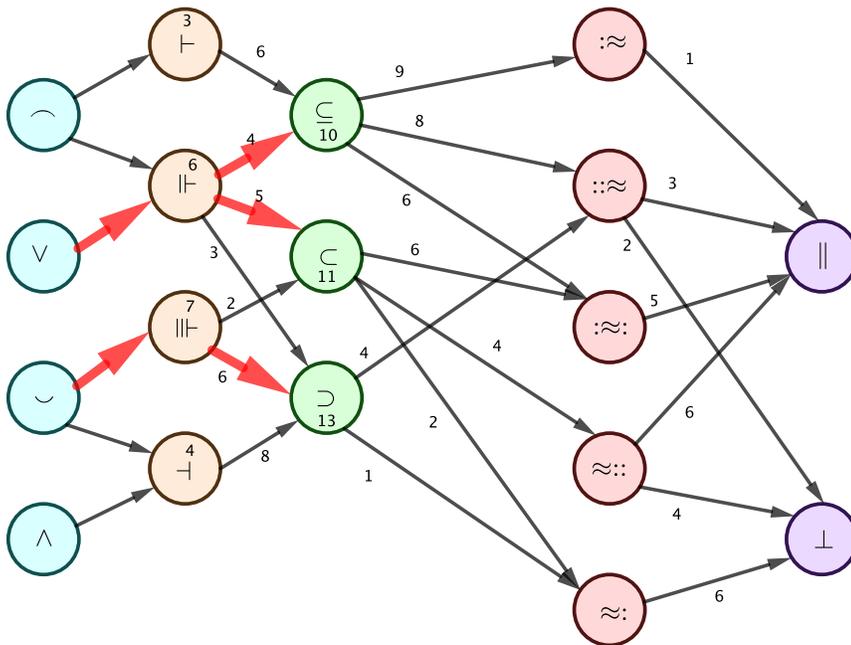
The algorithmic idea to solve this problem quickly is called dynamic programming. It is based on a general idea of systematically building the solution from small chunks to bigger and bigger pieces. If you remember (or write down) the partial results, these partial results can be used to calculate a solution without having to recompute these partial results. This problem also gives you a glimpse of contemporary machine translation. It may be somewhat surprising, but machine translation does not depend on a deep understanding of grammar rules. Rather, it works with enormous databases of texts in different languages, and simply put, looks for good matches, especially with digrams and trigrams (pairs or triplets of words that occur frequently).



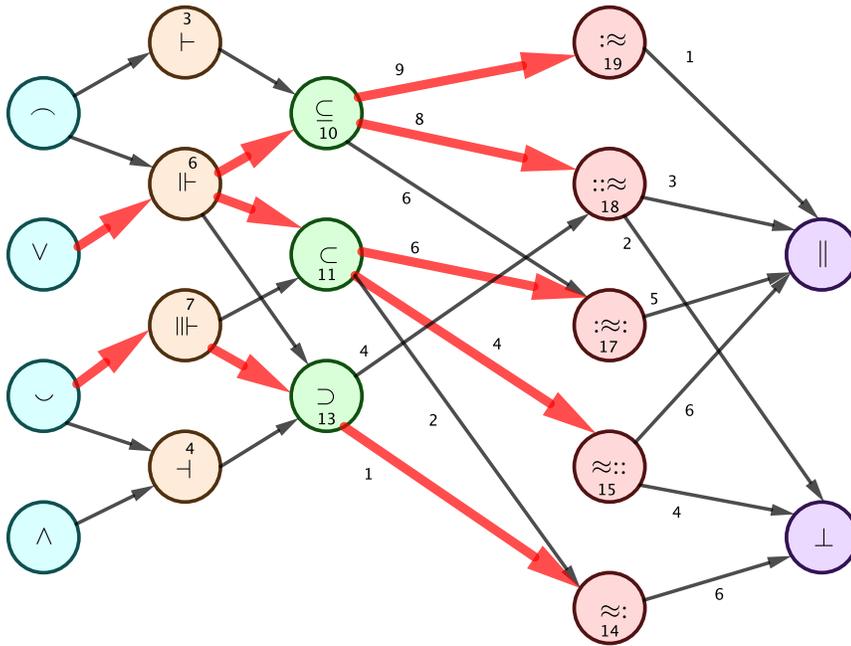
First Pass:



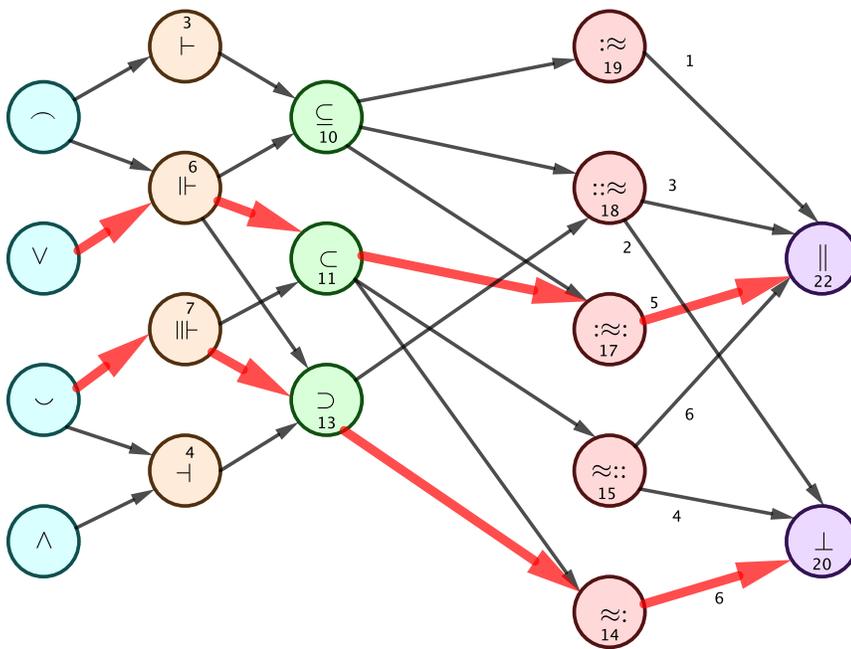
Second Pass:



Third Pass:



Fourth Pass:

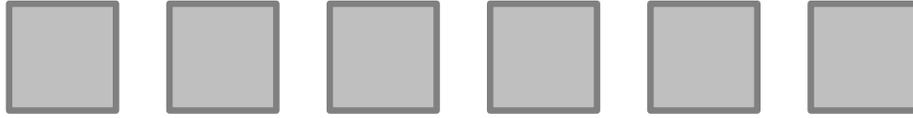




Problem of the Week

Problem E

Access Denied



A six-digit code is required to gain access to a secure building.

You know the following about the correct code:

- only the digits 1, 2, 3, 4, 5, and 6 are used in the correct code and each of these digits appears exactly once;
- the fifth digit in the code is not a 5;
- the sixth digit in the code is not a 6; and
- there is only one correct code.

What is the probability that you are able to enter the correct code?

Note: Often in counting problems, products such as $4 \times 3 \times 2 \times 1$ are encountered. This product can be written using factorial notation, as $4!$. In general, if n is a positive integer, then $n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1$.





Problem of the Week

Problem E and Solution

Access Denied

Problem

A six-digit code is required to gain access to a secure building. You know the following about the correct code: only the digits 1, 2, 3, 4, 5, and 6 are used in the correct code and each of these digits appears exactly once, the fifth digit in the code is not a 5, the sixth digit in the code is not a 6, and there is only one correct code. What is the probability that you are able to enter the correct code?

Solution

Solution 1

In this solution we will determine the number of possibilities directly, examining the possible cases.

1. Neither the 5 nor the 6 appear in the fifth or sixth positions.

There are four possible positions for the 5 and for each of these choices there are 3 possible positions for the 6. There are $3 \times 4 = 12$ ways to place the 5 and 6. Once placed, there are four positions left to fill with the four remaining digits. These can be placed in $4 \times 3 \times 2 \times 1$ or $4!$ ways. Therefore, there are $12 \times 4! = 12 \times 24 = 288$ possible numbers in which neither the 5 or the 6 are in the fifth or sixth positions.

2. The 6 appears in the fifth position and the 5 appears in sixth position.

There is 1 way to place the 5 and 6. The remaining positions can be filled in $4!$ or 24 ways.

3. The 6 is in the fifth position but the 5 is not in the sixth position.

There are 4 ways to place the 5. For each of these, there are $4!$ ways to place the remaining four digits. Therefore, there are $4 \times 4! = 96$ possible numbers in which 6 is in the fifth position, and 5 is not in the sixth position.

4. The 5 is in the sixth position but the 6 is not in fifth position.

There are 4 ways to place the 6. For each of these, there are $4!$ ways to place the remaining four digits. Therefore, there are $4 \times 4! = 96$ possible numbers in which 5 is in the sixth position, and 6 is not in the fifth position.

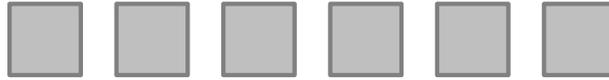
Therefore, there are $288 + 24 + 96 + 96 = 504$ different possible codes that satisfy the given conditions. The probability of entering the correct code is $\frac{1}{504}$.



Solution 2

In this solution we will determine the number of possibilities indirectly.

We will calculate the total possible number of codes using each of the digits 1, 2, 3, 4, 5, and 6 exactly once without considering the other restrictions. We will then subtract the codes which are not permitted.



The total number of different six-digit codes using the numbers 1, 2, 3, 4, 5, and 6 exactly once is $6!$ or 720. But this includes the following possibilities which are **not** permitted.

1. The 5 is in the fifth position and the 6 is in the sixth position.



The four remaining digits can be filled in $4!$ or 24 ways. There are 24 codes in which the 5 is in the fifth position and the 6 is in the sixth position.

2. The 5 is in the fifth position but the 6 is not in the sixth position.



Since the 6 cannot be in the sixth position, there are only four positions for the 6. For each of these 4 possible placements of the 6 there are $4!$ ways to place the remaining digits. There are $4 \times 4! = 4 \times 24 = 96$ codes in which the 5 is in the fifth position but the 6 is not in the sixth position.

3. The 6 is in the sixth position but the 5 is not in the fifth position.



Since the 5 cannot be in the fifth position, there are only four positions for the 5. For each of these 4 possible placements of the 5 there are $4!$ ways to place the remaining digits. There are $4 \times 4! = 4 \times 24 = 96$ codes in which the 6 is in the sixth position but the 5 is not in the fifth position.

Therefore, there are $720 - 24 - 96 - 96 = 504$ different possible codes that satisfy the given conditions. The probability of entering the correct code is $\frac{1}{504}$.



Problem of the Week

Problem E

1 , 2 , 3

In Canada, a loonie is a dollar coin. A two player game has players alternating turns by removing loonies from three piles. At the beginning of the game, the first pile contains 1 loonie, the second pile contains 2 loonies and the third pile contains 3 loonies. Players take turns removing one or more coins from any **one** pile. A player can remove from one coin to all of the coins in a particular pile on their turn. The player who removes the last coin as part of their turn wins the game.

Jessie and Jason play this game. Jessie goes first. Jason claims that no matter what Jessie does on his turn, that he can win by following his strategy.

Describe Jason's winning strategy. (To discover Jason's winning strategy, play the game many times.)





Problem of the Week

Problem E and Solution

1 , 2 , 3

Problem

In Canada, a loonie is a dollar coin. A two player game has players alternating turns by removing loonies from three piles. At the beginning of the game, the first pile contains 1 loonie, the second pile contains 2 loonies and the third pile contains 3 loonies. Players take turns removing one or more coins from any **one** pile. A player can remove from one coin to all of the coins in a particular pile on their turn. The player who removes the last coin as part of their turn wins the game. Jessie and Jason play this game. Jessie goes first. Jason claims that no matter what Jessie does on his turn, that he can win by following his strategy. Describe Jason's winning strategy.

Solution

Jessie only has 6 possible first moves. He can remove 1 coin from pile one, two or three; or 2 coins from pile two or three; or all 3 coins from pile three. Jason realized that no matter what Jessie does first, he can remove coins in some way so that when his first turn is over there are two equal size piles.

He noticed that this can happen in two different ways.

- On Jessie's first turn he chooses to remove all the coins from one of the three piles. This always results in two different size piles. Jason then removes enough coins from the larger pile so that when his turn is complete, the two remaining piles contain the same number of coins.

Start of Game	Jessie's first move	result	Jason's first move	result
1 2 3	remove pile one	0 2 3	remove 1 coin from pile three	0 2 2
1 2 3	remove pile two	1 0 3	remove 2 coins from pile three	1 0 1
1 2 3	remove pile three	1 2 0	remove 1 coin from pile two	1 1 0

- On Jessie's first turn he chooses to remove coins from pile 2 or pile 3 in such a way that after his turn three piles remain. Jason notices that two of the remaining piles are always the same size and the third pile is a different size. So on his first turn Jason removes all of the coins from the pile which has a different number of coins than the other two piles. After his turn, the two remaining piles contain the same number of coins.

Start of Game	Jessie's first move	result	Jason's first move	result
1 2 3	remove 1 coin from pile two	1 1 3	remove all of pile three	1 1 0
1 2 3	remove 1 coin from pile three	1 2 2	remove all of pile one	0 2 2
1 2 3	remove 2 coins from pile three	1 2 1	remove all of pile two	1 0 1

So, after Jason's first turn, he leaves Jessie with only two piles, each containing the same number of coins. More specifically, after Jason's first turn there will be either two piles with 2 coins in each or two piles with 1 coin in each.



In the four cases where, after Jason's first turn, the two remaining piles each contain 1 coin, Jessie must remove one of the piles on his second turn. Then on his second turn Jason removes the final coin and wins. The table illustrates an example of how a game plays out after Jason's first turn produces 1 coin in pile one, no coins in pile two and 1 coin in pile three.

	Pile One	Pile Two	Pile Three
After Jason's first turn	1	0	1
Jessie removes pile one	0	0	1
Jason removes the final coin for the win	0	0	0

In the two cases where there are only two piles, each containing 2 coins, Jessie has two choices for his second turn. If Jessie removes both coins from one of the piles, Jason responds by removing both coins from the remaining pile and wins. If Jessie removes 1 coin from one of the piles on his second turn, Jason uses his second turn to take 1 coin from the other pile. He always wants to leave two equal size piles by the time any turn is completed. After his second turn there would be two piles, each containing 1 coin. Jessie must then remove one of the piles on his third turn. Jason then removes the final coin on his third turn and wins. The table below illustrates an example of how a game plays out after Jason's first turn produces no coins in pile one, 2 coins in pile two and 2 coins in pile three.

	Pile One	Pile Two	Pile Three
After Jason's first turn	0	2	2
Jessie removes one coin from pile three	0	2	1
Jason removes 1 coin from pile two to even out the piles	0	1	1
Jessie removes pile two	0	0	1
Jason removes the final coin from pile three for the win	0	0	0

So Jason's strategy is fairly straight forward. After Jessie's first turn, he should remove coin(s) from a pile so that only two piles remain and each pile contains the same number of coins. On his second and possible third turn, he should copy what Jessie did on his previous turn, but to the pile he did not touch. Once Jason's strategy is discovered, Jessie will no longer want to play this game.

For further Investigation

Jessie proposes two new versions of the coin game. For these new versions, does either player have a winning strategy? Jessie wants to see if what he learned from the first game can be applied to either of these new games.

1. Suppose there are just two piles to start, one pile has 13 coins and the other has 17 coins. On any turn a player can remove any number of coins from one of the piles. Players alternate turns and the player to remove the last coin wins.
2. Suppose there are three piles of coins to start, one pile has 2 coins, a second pile has 4 coins and the third pile has 5 coins. On any turn a player can remove any number of coins from one of the piles. Players alternate turns and the player to remove the last coin wins.



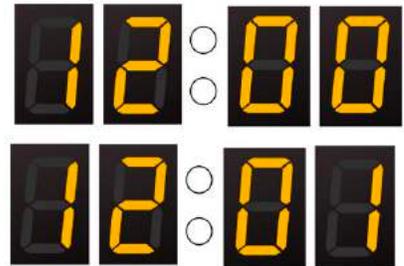
Problem of the Week

Problem E

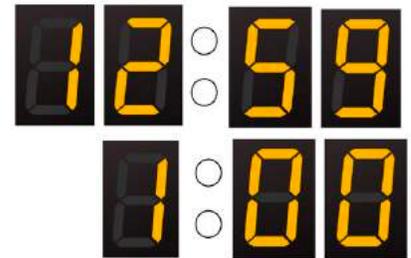
Around the Clock

The digits are displayed on a clock in the following way. Each of the four digit spots consists of seven LEDs (Light-Emitting Diodes) which are turned off or on depending on the digit to be displayed.

When the time changes from “12:00” to “12:01”, the first three digits of the time remain in their same state but four of the LEDs in the final digit change from on to off.



Many more things happen when the clock changes from “12:59” to “1:00”. In the first digit spot, the two LEDs used to display the 1 turn off. In the second digit spot, four of the LEDs that are on to make the 2 turn off and another LED turns on. In the third digit spot, to change from a 5 to a 0, two LEDs need to turn on and one LED needs to turn off. In the fourth digit spot, to change from a 9 to a 0, one LED must turn off and one LED must turn on. In total, to change from “12:59” to “1:00”, a total of $2 + 5 + 3 + 2 = 12$ changes occur.



If the initial time displayed on the clock is “12:00”, how many changes have occurred once the clock displays “12:00” again?

Each of the digits is shown below for your reference.





Problem of the Week

Problem E and Solution

Around the Clock

Problem

The digits are displayed on a clock in the following way. Each of the four digit spots consists of seven LEDs (Light-Emitting Diodes) which are turned off or on depending on the digit to be displayed. When the time changes from “12:00” to “12:01”, the first three digits of the time remain in their same state but four of the LEDs in the final digit change from on to off. Many more things happen when the clock changes from “12:59” to “1:00”. In the first digit spot, the two LEDs used to display the 1 turn off. In the second digit spot, four of the LEDs that are on to make the 2 turn off and another LED turns on. In the third digit spot, to change from a 5 to a 0, two LEDs need to turn on and one LED needs to turn off. In the fourth digit spot, to change from a 9 to a 0, one LED must turn off and one LED must turn on. In total, to change from “12:59” to “1:00”, a total of $2 + 5 + 3 + 2 = 12$ changes occur. If the initial time displayed on the clock is “12:00”, how many changes have occurred once the clock displays “12:00” again?

Solution



This problem has many moving parts. To start, let’s examine all of the possible changes that can take place in any of the four digit spots. The changes are summarized in the chart below.

Example Time	In Changing	# of LEDs changing from OFF to ON	# of LEDs changing from ON to OFF	# of LEDs remaining in the same state	Total # of changes in state
9:59 to 10:00	blank to 1	2	0	5	2
2:50 to 2:51	0 to 1	0	4	3	4
12:59 to 1:00	1 to blank	0	2	5	2
1:59 to 2:00	1 to 2	4	1	2	5
12:59 to 1:00	2 to 1	1	4	2	5
1:52 to 1:53	2 to 3	1	1	5	2
1:53 to 1:54	3 to 4	1	2	4	3
1:54 to 1:55	4 to 5	2	1	4	3
3:59 to 4:00	5 to 0	2	1	4	3
12:45 to 12:46	5 to 6	1	0	6	1
12:46 to 12:47	6 to 7	1	4	2	5
12:47 to 12:48	7 to 8	4	0	3	4
12:48 to 12:49	8 to 9	0	1	6	1
3:49 to 3:50	9 to 0	1	1	5	2



1. Consider the changes in the fourth digit from the left that take place in 12 hours.

Every 10 minutes the fourth digit changes from

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 0$$

By referring to the table, there are a total of $4 + 5 + 2 + 3 + 3 + 1 + 5 + 4 + 1 + 2$ or 30 changes in the fourth digit every 10 minutes. Every hour, the 30 changes happen 6 times or a total of $6 \times 30 = 180$ times. In 12 hours, the 180 changes take place 12 times for a total of 2160 changes in the fourth digit.

2. Consider the changes in the third digit from the left that take place in 12 hours.

Every hour the third digit changes from

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 0$$

By referring to the table, there are a total of $4 + 5 + 2 + 3 + 3 + 3$ or 20 changes in the third digit every hour. Since we are considering 12 hours, there are a total of 12×20 or 240 changes in the third digit.

3. Consider the changes in the second digit from the left that take place in 12 hours.

The second digit changes from

$$2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 0 \rightarrow 1 \rightarrow 2$$

By referring to the table, there are a total of $5 + 5 + 2 + 3 + 3 + 1 + 5 + 4 + 1 + 2 + 4 + 5$ or 40 changes in the second digit every 12 hours.

4. Consider the changes in the leftmost digit that take place in 12 hours.

The leftmost digit changes from

$$1 \rightarrow \text{blank} \rightarrow 1$$

By referring to the table, there are a total of $2 + 2$ or 4 changes in the leftmost digit every 12 hours.

Combining the results from the four cases, there are a total of $2160 + 240 + 40 + 4$ or 2444 changes in the state of the LEDs in the 12 hour period.



Problem of the Week

Problem E

Great POTW Bake-Off

Pepper, Belle, Emerson, and Ralphie have just competed in POTW's Annual Bake-Off. Pepper, Belle, Emerson, and Ralphie each came in first, second, third, or fourth in the competition. There were no ties. The baker who came in first placed highest and the baker who came in last placed lowest.

Each baker created a different variety of cookie. One baked ginger snaps, one baked sugar cookies, one baked lemon drops, and one baked peppermint crunch cookies.

Each baker also presented their baked goods to the judges on a different coloured plate. One on a white plate, one on a red plate, one on a brown plate, and one on a silver plate.

Using the following clues, determine which baker came in first, second, third, and fourth, which baker baked which variety of cookie, and on which plate they presented their baked goods to the judges.

1. The ginger snaps placed higher than the cookies baked by Emerson, but lower than the cookies presented on the brown plate.
2. The cookies baked by Pepper placed directly below the sugar cookies.
3. The cookies presented on the silver plate placed directly above the cookies presented on the red plate. Belle did not present her cookies on the silver plate.
4. The lemon drops were presented on the white plate and did not come in last place.
5. Ralphie did not come in third. The cookies presented on the white plate were not baked by Ralphie.

You may find the table on the following page helpful in organizing your solution to this problem.



	First	Second	Third	Fourth	White	Red	Brown	Silver	Ginger Snaps	Sugar Cookies	Lemon Drops	Peppermint Crunch
Pepper												
Belle												
Emerson												
Ralphie												
Ginger Snaps												
Sugar Cookies												
Lemon Drops												
Peppermint Crunch												
White												
Red												
Brown												
Silver												





Problem of the Week

Problem E and Solution

Great POTW Bake-Off

Problem

Pepper, Belle, Emerson, and Ralphie have just competed in POTW's Annual Bake-Off. Pepper, Belle, Emerson, and Ralphie each came in first, second, third, or fourth in the competition. There were no ties. The baker who came in first placed highest and the baker who came in last placed lowest.

Each baker created a different variety of cookie. One baked ginger snaps, one baked sugar cookies, one baked lemon drops, and one baked peppermint crunch cookies.

Each baker also presented their baked goods to the judges on a different coloured plate. One on a white plate, one on a red plate, one on a brown plate, and one on a silver plate.

Using the following clues, determine which baker came in first, second, third, and fourth, which baker baked which variety of cookie, and on which plate they presented their baked goods to the judges.

1. The ginger snaps placed higher than the cookies baked by Emerson, but lower than the cookies presented on the brown plate.
2. The cookies baked by Pepper placed directly below the sugar cookies.
3. The cookies presented on the silver plate placed directly above the cookies presented on the red plate. Belle did not present her cookies on the silver plate.
4. The lemon drops were presented on the white plate and did not come in last place.
5. Ralphie did not come in third. The cookies presented on the white plate were not baked by Ralphie.

Solution

We will present the answer first for those who want to just check their final answer. The solution that follows gives one possible approach which leads to a correct set of conclusions.

- Belle presented the lemon drops on the white plate and came in first.
- Ralphie presented the sugar cookies on the brown plate and came in second.
- Pepper presented the ginger snaps on the silver plate and came in third.
- Emerson presented the peppermint crunch cookies on the red plate and came in fourth.



Solution

In our solution, we will go through each clue and update the table based on the information in the clue. We will put an X in a cell if the combination indicated by the row and column for that cell is not possible, or a ✓ if it must be true.

From clue (1), we can determine that the ginger snaps did not come in fourth place and that the cookies made by Emerson did not come in first, since the ginger snaps placed higher than the cookies made by Emerson, and that the ginger snaps were not made by Emerson. We can place an X in each of the corresponding cells.

From this clue we can also determine that the ginger snaps did not come in first place and the cookies on the brown plate did not come in fourth since the ginger snaps placed lower than the cookies presented on the brown plate. We can also determine that the ginger snaps were not presented on the brown plate and that the cookies baked by Emerson were not presented on the brown plate. We can place an X in each of the corresponding cells and the table is updated to the right.

	First	Second	Third	Fourth	White	Red	Brown	Silver	Ginger Snaps	Sugar Cookies	Lemon Drops	Peppermint Crunch
Pepper												
Belle												
Emerson	X						X		X			
Ralphie												
Ginger Snaps	X			X			X					
Sugar Cookies												
Lemon Drops												
Peppermint Crunch												
White												
Red												
Brown				X								
Silver												

From clue (2), we can determine that the cookies baked by Pepper did not come in first place and that the sugar cookies did not come in fourth place. We can also determine that Pepper did not bake the sugar cookies. We can place an X in each of the corresponding cells.

From clue (3), we can determine that the cookies presented on the silver plate did not come in fourth place and that the cookies presented on the red plate did not come in first.

	First	Second	Third	Fourth	White	Red	Brown	Silver	Ginger Snaps	Sugar Cookies	Lemon Drops	Peppermint Crunch
Pepper	X									X		
Belle								X				
Emerson	X						X		X			
Ralphie												
Ginger Snaps	X			X			X					
Sugar Cookies				X								
Lemon Drops												
Peppermint Crunch												
White												
Red	X											
Brown				X								
Silver				X								

We know that Belle did not present her cookies on the silver plate. We can place an X in each of the corresponding cells. The table is updated above.



From clue (4), we know that the lemon drops were presented on the white plate. We can place a ✓ in the corresponding cell. We also know that the lemon drops and the white plate did not come in last place. We can place an X in each of the corresponding cells. The table to the right contains the updated information.

	First	Second	Third	Fourth	White	Red	Brown	Silver	Ginger Snaps	Sugar Cookies	Lemon Drops	Peppermint Crunch
Pepper	X									X		
Belle								X				
Emerson	X						X		X			
Ralphie												
Ginger Snaps	X			X			X					
Sugar Cookies				X								
Lemon Drops				X	✓							
Peppermint Crunch												
White				X								
Red	X											
Brown				X								
Silver				X								

Since no two types of cookie can be presented on the same plate, we can place an X in each of the cells corresponding to the ginger snaps, sugar cookies, and peppermint crunch cookies being presented on the white plate.

Similarly, only one type of cookie can be presented on a plate so we can place an X in each of the cells corresponding to the lemon drops being presented on the red, brown, or silver plate.

We can also see that the peppermint crunch cookies must come in fourth place and that the cookies presented on the red plate also come in fourth place. We can place a ✓ in each of the corresponding cells. Using similar logic as above we can also place an X in each of the cells corresponding to the peppermint crunch cookies coming in first, second or third and the red cookies coming in second or third.

	First	Second	Third	Fourth	White	Red	Brown	Silver	Ginger Snaps	Sugar Cookies	Lemon Drops	Peppermint Crunch
Pepper	X									X		
Belle								X				
Emerson	X						X		X			
Ralphie												
Ginger Snaps	X			X	X	X	X					
Sugar Cookies				X	X	X						
Lemon Drops				X	✓	X	X	X				
Peppermint Crunch	X	X	X	✓	X	✓	X	X				
White				X								
Red	X	X	X	✓								
Brown				X								
Silver				X								

We can also determine that since the peppermint crunch cookies come in fourth place and the cookies presented on the red plate come in fourth place, that the peppermint crunch cookies must be presented on the red plate. We can add the corresponding ✓'s and X's and the table is updated above.

We now know that the sugar cookies were presented on the brown plate and that the ginger snaps were presented on the silver plate. We can add the corresponding ✓'s and X's.

Now, let's take a minute to look back at what we know so far in order to eliminate some possibilities. Since we know that the ginger snaps were presented on the silver plate, and the ginger snaps cannot come in first place, this tells us that the silver plate cannot come in first place.

Then, since Pepper did not bake the sugar cookies and the sugar cookies were presented on the brown plate, then Pepper did not present her cookies on the brown plate.



From this clue we can also determine that Belle did not come in third since she did not present her cookies on the silver plate. We can add the corresponding \checkmark 's and X's. The table is updated above.

Now look at clue (1). Since the ginger snaps placed higher than the cookies baked by Emerson, and the ginger snaps came in third place, then Emerson must have baked the cookies that came in fourth place. Therefore, Emerson must have baked the peppermint crunch cookies and presented them on the red plate. We can add the corresponding \checkmark 's and X's and update the table as follows.

	First	Second	Third	Fourth	White	Red	Brown	Silver	Ginger Snaps	Sugar Cookies	Lemon Drops	Peppermint Crunch
Pepper	X			X		X	X			X		X
Belle			X	X		X		X	X			X
Emerson	X	X	X	\checkmark	X	\checkmark	X	X	X	X	X	\checkmark
Ralphie			X	X	X	X					X	X
Ginger Snaps	X	X	\checkmark	X	X	X	X	\checkmark				
Sugar Cookies			X	X	X	X	\checkmark	X				
Lemon Drops			X	X	\checkmark	X	X	X				
Peppermint Crunch	X	X	X	\checkmark	X	\checkmark	X	X				
White			X	X								
Red	X	X	X	\checkmark								
Brown			X	X								
Silver	X	X	\checkmark	X								

We now see that Pepper came in third place. This also tells us that Pepper baked the ginger snaps and presented them on the silver plate. We can add the corresponding \checkmark 's and X's and update the table as shown to the right.

	First	Second	Third	Fourth	White	Red	Brown	Silver	Ginger Snaps	Sugar Cookies	Lemon Drops	Peppermint Crunch
Pepper	X	X	\checkmark	X	X	X	X	\checkmark	\checkmark	X	X	X
Belle			X	X		X		X	X			X
Emerson	X	X	X	\checkmark	X	\checkmark	X	X	X	X	X	\checkmark
Ralphie			X	X	X	X		X	X		X	X
Ginger Snaps	X	X	\checkmark	X	X	X	X	\checkmark				
Sugar Cookies			X	X	X	X	\checkmark	X				
Lemon Drops			X	X	\checkmark	X	X	X				
Peppermint Crunch	X	X	X	\checkmark	X	\checkmark	X	X				
White			X	X								
Red	X	X	X	\checkmark								
Brown			X	X								
Silver	X	X	\checkmark	X								

We can see now that Belle presented her cookies on the white plate and baked the lemon drops. So, Ralphie must have presented his cookies on the brown plate and baked the sugar cookies. We can add the corresponding \checkmark 's and X's.

To determine who came in first place and who came in second, we need to analyze clue (2). From this clue, since the cookies baked by Pepper placed directly below the sugar cookies and Pepper came in third place, then the sugar cookies must have come in second place. Therefore, Ralphie came in second place. So, Belle came in first place. We can add the corresponding \checkmark 's and X's and complete the table as follows.



	First	Second	Third	Fourth	White	Red	Brown	Silver	Ginger Snaps	Sugar Cookies	Lemon Drops	Peppermint Crunch
Pepper	X	X	✓	X	X	X	X	✓	✓	X	X	X
Belle	✓	X	X	X	✓	X	X	X	X	X	✓	X
Emerson	X	X	X	✓	X	✓	X	X	X	X	X	✓
Ralphie	X	✓	X	X	X	X	✓	X	X	✓	X	X
Ginger Snaps	X	X	✓	X	X	X	X	✓				
Sugar Cookies	X	✓	X	X	X	X	✓	X				
Lemon Drops	✓	X	X	X	✓	X	X	X				
Peppermint Crunch	X	X	X	✓	X	✓	X	X				
White	✓	X	X	X								
Red	X	X	X	✓								
Brown	X	✓	X	X								
Silver	X	X	✓	X								

Reading the results from the table:

- Pepper presented the ginger snaps on the silver plate and came in third.
- Belle presented the lemon drops on the white plate and came in first.
- Emerson presented the peppermint crunch cookies on the red plate and came in fourth.
- Ralphie presented the sugar cookies on the brown plate and came in second.



Problem of the Week

Problem E

More Digit Swapping

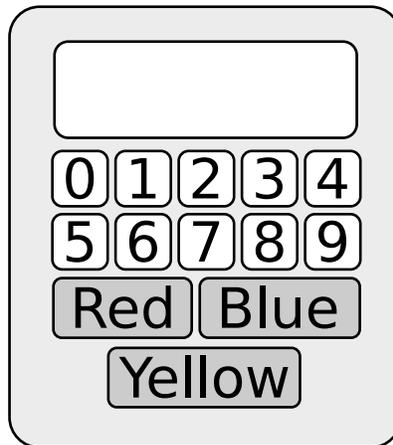
Ali programs three buttons in a machine to swap digits in a 4-digit integer.

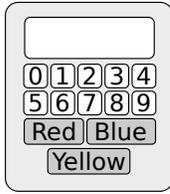
Red button: swaps the thousands and tens digits

Blue button: swaps the thousands and hundreds digits

Yellow button: swaps the hundreds and units (ones) digits

Ali types the integer 1234 into the machine. Using only the Red, Blue, and Yellow buttons, determine all outputs she can produce with exactly 5 more button presses that she cannot produce using fewer than 5 more button presses.





Problem of the Week

Problem E and Solution

More Digit Swapping

Problem

Ali programs three buttons in a machine to swap digits in a 4-digit integer.

Red button: swaps the thousands and tens digits

Blue button: swaps the thousands and hundreds digits

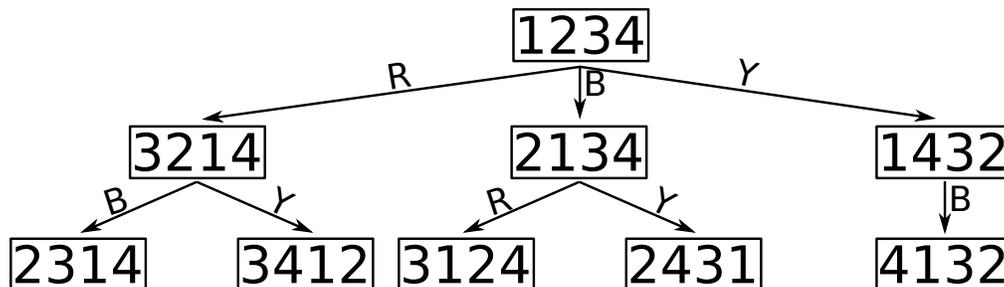
Yellow button: swaps the hundreds and units (ones) digits

Ali types the integer 1234 into the machine. Using only the Red, Blue, and Yellow buttons, determine all outputs she can produce with exactly 5 more button presses that she cannot produce using fewer than 5 more button presses.

Solution

We make a tree diagram to show all the possible outputs. We create this tree diagram one row at a time, moving from left to right. When we get a number that already exists in the tree, we do not write it down so our tree does not contain any duplicates. This will ensure that the tree shows only the shortest path to reach each of the possible different outputs.

We can then use this tree diagram to find all the outputs that require exactly 5 more button presses to produce and cannot be produced in less.

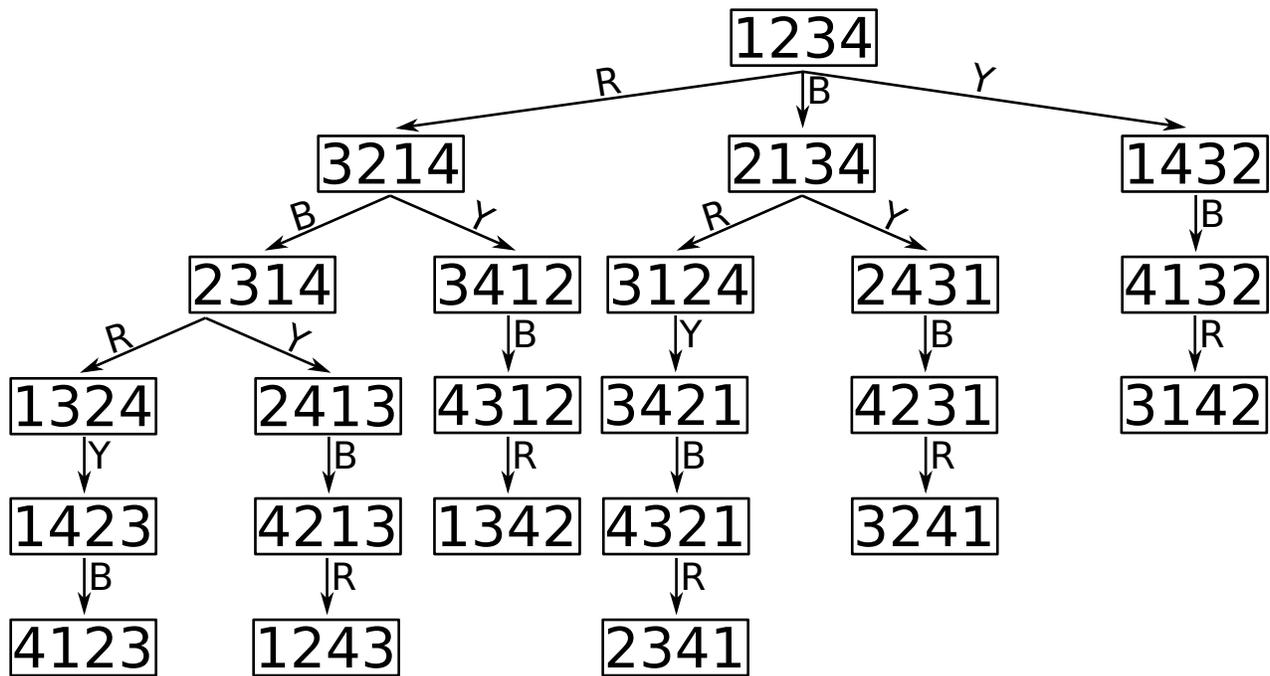


Notice that we did not write down the output for pressing the Red button twice. This is because we would have ended up with 1234, which the number we were at before the first Red button push, and so is already in our tree. The same argument applies for pressing any button twice in a row.

Notice also that we did not write down the output for pressing the Yellow button followed by the Red button. This is because we would have ended up with 3412, which is already in our tree.



We continue with the tree diagram, stopping after we have gone through all possible outputs from pressing 5 buttons.



Therefore, the outputs which can be produced by exactly five more button presses and cannot be produced with fewer are 4123, 1243, and 2341.

Extension: For the input of 1234, which output requires the most presses of the Red, Blue, and Yellow buttons to produce?



Problem of the Week

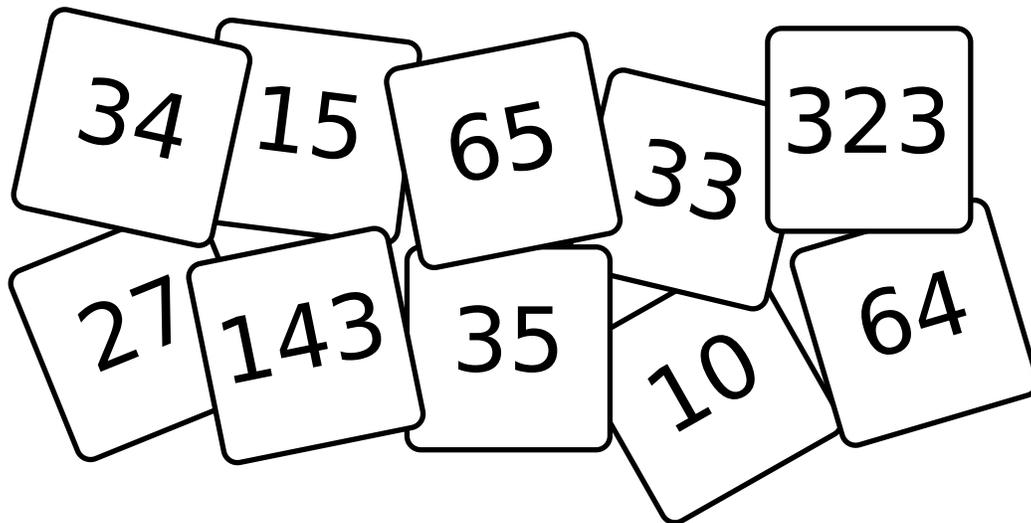
Problem E

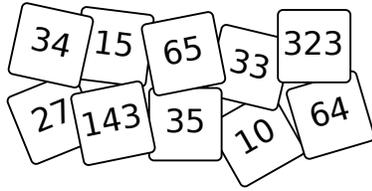
The Factor Flip

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up.

Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up.

List all the possible orders in which Dani can flip the cards so that all cards get flipped over.





Problem of the Week

Problem E and Solution

The Factor Flip

Problem

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up. Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up. List all the possible orders in which Dani can flip the cards so that all cards get flipped over.

Solution

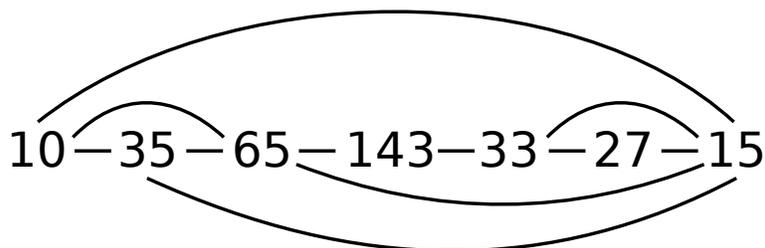
We will start by writing down the prime factors for each of the numbers on the cards.

Number	Prime Factors	Number	Prime Factors
10	2,5	35	5,7
15	3,5	64	2
27	3	65	5,13
33	3,11	143	11,13
34	2,17	323	17,19

Notice the number 323 shares a prime factor with only the number 34. That means we must start (or end) with 323.

If we start with 323, then the next number must be 34. From 34, the next number could be 64 or 10. If the next number were 10, then in order to eventually flip over the 64 card, the 64 must follow the 10, and at this point no more cards can be flipped. Therefore, in order to flip all the cards, the first four numbers flipped must be 323, 34, 64, and then 10.

From this point, we can draw lines between numbers that share prime factors to create the following diagram.





We now just need to figure out the number of paths through the diagram, starting at 10 that use each number exactly once.

After 10, the next number can be 35, 65, or 15. In each case, we carefully trace through all possible paths.

Case 1: The number 35 follows the number 10. That gives us the following five possible paths.

35, 15, 65, 143, 33, 27

35, 15, 27, 33, 143, 65

35, 65, 15, 27, 33, 143

35, 65, 143, 33, 27, 15

35, 65, 143, 33, 15, 27

Case 2: The number 65 follows the number 10. That gives us the following two possible paths.

65, 143, 33, 27, 15, 35

65, 35, 15, 27, 33, 143

Case 3: The number 15 follows the number 10. That gives us the following two possible paths.

15, 27, 33, 143, 65, 35

15, 35, 65, 143, 33, 27

Starting with the card numbered 323, we have found that there is a total of nine orders for flipping the cards:

323, 34, 64, 10, 35, 15, 65, 143, 33, 27

323, 34, 64, 10, 35, 15, 27, 33, 143, 65

323, 34, 64, 10, 35, 65, 15, 27, 33, 143

323, 34, 64, 10, 35, 65, 143, 33, 27, 15

323, 34, 64, 10, 35, 65, 143, 33, 15, 27

323, 34, 64, 10, 65, 143, 33, 27, 15, 35

323, 34, 64, 10, 65, 35, 15, 27, 33, 143

323, 34, 64, 10, 15, 27, 33, 143, 65, 35

323, 34, 64, 10, 15, 35, 65, 143, 33, 27

Each of these can be reversed, so the total number of possible orders is 18.

Therefore, there are 18 orders in which Dani can flip the cards so that all cards get flipped over. They are the 9 orders listed above and the reverse of each.