Problem of the Week
Problem D and Solution

ELEVEN

Problem

The word ELEVEN contains four different letters, $E$, $L$, $V$, and $N$. Each letter in the word ELEVEN is assigned a different integer value between 0 and 9, inclusive, to create a six-digit positive integer. If, for example, $E = 4$, $L = 5$, $V = 6$, and $N = 1$, then the resulting number is 454641. The choice of these digit values for the letters in the word ELEVEN is particularly interesting since the resulting number is divisible by 11.

1. Determine the values of $E$, $L$, $V$, and $N$ which make ELEVEN as large as possible subject to the condition that ELEVEN must be divisible by 11; and

2. determine the values of $E$, $L$, $V$, and $N$ which make ELEVEN as small as possible, subject to the condition that ELEVEN must be divisible by 11. (Also remember, $E \neq L \neq V \neq N$.)

Solution

1. Find the largest six-digit positive integer ELEVEN which is divisible by 11 such that $E \neq L \neq V \neq N$.

   In order to make the number as large as possible, we make the leftmost digit as large as possible. In this case, we must make the ten-thousand’s digit $E = 9$. By letting $E = 9$, the resulting number looks like $9L9V9N$. The sum of the digits in the odd positions would be $9 + 9 + 9 = 27$. We want $27 - (L + V + N)$ to be a multiple of 11.

   The next digit to replace is $L$. Again, we will set $L$ equal to the largest possible available number, namely 8. The number now looks like $989V9N$. We want $27 - (8 + V + N)$ to be a multiple of 11.

   Set $V = 7$, the next largest available remaining value. The number looks like $98979N$. We want $27 - (8 + 7 + N) = 12 - N$ to be a multiple of 11.

   We select the largest value from 0 to 6 so that $12 - N$ is a multiple of 11. The only value that satisfies this is $N = 1$ and the resulting number is $989791$. This number equals $11 \times 89981$ and is therefore divisible by 11.
2. Find the smallest six-digit positive integer ELEVEN which is divisible by 11 such that \( E \neq L \neq V \neq N \).

In order to make the number as small as possible, we make the leftmost digit as small as possible. In this case, we must make the ten-thousand’s digit \( E = 1 \). If \( E = 0 \), then we no longer have a six-digit number. The number \( 0L0V0N \) is a five-digit number. By letting \( E = 1 \), the resulting number looks like \( 1L1V1N \). The sum of the digits in the odd positions would be \( 1 + 1 + 1 = 3 \). We want \( 3 - (L + V + N) \) to be a multiple of 11.

The next digit to replace is \( L \). Again we will set \( L \) equal to the smallest possible available number, namely 0. The number now looks like 101V1N. We want \( 3 - (0 + V + N) = 3 - (V + N) \) to be a multiple of 11. If \( V + N = 3 \) then \( 3 - (V + N) = 0 \) and 101V1N would be a multiple of 11. We could do this if the digits 0 and 3 or 1 and 2 were assigned to \( V \) and \( N \) in some order. But, in either case, one of the digits has already been used. It is not possible to get ELEVEN to be such that \( E = 1 \), \( L = 0 \), and \( V + N = 3 \) so that all of the unknowns have distinct values.

We might be tempted to give up and assign a different value to \( E \). However, we must remember that we are looking for values of \( V \) and \( N \) so that \( 3 - (V + N) \) is a multiple of 11. If \( V + N = 14 \), then
\[
3 - (V + N) = 3 - 14 = -11,
\]
which is a multiple of 11. We want to find a combination of available values from 2 to 9 such that \( V + N = 14 \) and \( V \) is as small as possible. Many combinations exist, namely, 5 and 9, 6 and 8, 7 and 7, 8 and 6, and 9 and 5. We obtain the smallest possible value for \( V \) when \( V = 5 \) and \( N = 9 \). It then follows that the smallest number is 101519. Since this number is 11 \( \times \) 9229 = 101519, it is divisible by 11.

Therefore, when \( E = 9 \), \( L = 8 \), \( V = 7 \), and \( N = 1 \), ELEVEN becomes 989791. This is the largest six-digit number satisfying the conditions of the problem.

When \( E = 1 \), \( L = 0 \), \( V = 5 \), and \( N = 9 \), ELEVEN becomes 101519. This is the smallest six-digit number satisfying the conditions of the problem.