Problem

Four cards each have a different positive integer printed on one side. Also, one card is purple, one card is blue, one card is green, and one card is yellow. (For those without a colour printer, the colour of the card is printed below the card.) The four cards are placed on a table so that the numbered side is face down. Students look at the numbers on exactly three of the cards and then state the sum. One student looked under the purple, blue and green cards and stated that the sum was 14. A second student looked under the blue, green and yellow cards and stated that the sum was 18. Finally, a third student looked under the purple, green and yellow cards and stated that the sum was 19. Determine all possibilities for the positive integers on the purple, blue, green and yellow cards.

Solution

Solution 1

Let \( p, b, g, \) and \( y \) represent the positive integers written on the purple, blue, green and yellow cards, respectively. It is given that \( p \neq b \neq g \neq y \).

It is also given that
\[
\begin{align*}
p + b + g &= 14 \quad (1) \\
b + g + y &= 18 \quad (2) \\
p + g + y &= 19 \quad (3)
\end{align*}
\]

\((2) - (1) \) gives \( y - p = 4 \) or equivalently, \( y = p + 4 \) \( (4) \)

\((3) - (2) \) gives \( p - b = 1 \) or equivalently, \( b = p - 1 \) \( (5) \)

Since \( p, b, g \) and \( y \) are all different positive integers, let’s look at the possible values of \( p \) and calculate the values of \( b, g \) and \( y \) in each case:

<table>
<thead>
<tr>
<th>Purple Card</th>
<th>Blue Card</th>
<th>Green Card</th>
<th>Yellow Card</th>
<th>Possible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( b = p - 1 ) ( (5) )</td>
<td>( g = 14 - p - b ) (from ( (1) ))</td>
<td>( y = p + 4 ) ( (4) )</td>
<td>Yes or No</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>13</td>
<td>5</td>
<td>No ( (b \neq 0) )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>11</td>
<td>6</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>Yes</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>No ( (p = g) )</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>Yes</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>1</td>
<td>11</td>
<td>Yes</td>
</tr>
</tbody>
</table>

We can stop here, because if \( p > 7 \) then \( g < 0 \), which is not valid.

Therefore, there are 5 different possibilities for the numbers on the purple, blue, green and yellow cards, respectively.

They are \( (p, b, g, y) = (2, 1, 11, 6), (3, 2, 9, 7), (4, 3, 7, 8), (6, 5, 3, 10), (7, 6, 1, 11). \)
Solution 2

This solution is very similar to solution 1. The key difference is that, in this solution, we find an expression for each of \(b\), \(g\), and \(y\), in terms of \(p\). We then determine the smallest and largest possible values for \(p\). Based on these values we calculate the values of the other unknowns and determine the validity of each possibility.

Let \(p\), \(b\), \(g\), and \(y\) represent the positive integers written on the purple, blue, green and yellow cards, respectively. It is given that \(p \neq b \neq g \neq y\).

It is also given that

\[
egin{align*}
p + b + g &= 14 \quad (1) \\
b + g + y &= 18 \quad (2) \\
p + g + y &= 19 \quad (3)
\end{align*}
\]

\((2) - (1)\) gives \(y - p = 4\) and \(y = p + 4\) follows. \((4)\)

\((3) - (2)\) gives \(p - b = 1\) and \(b = p - 1\) follows. \((5)\)

Substitute for \(y\) from \((4)\) and \(b\) from \((5)\) into \((2)\) to get \(g\) in terms of \(p\).

\[
egin{align*}
b + g + y &= 18 \\
(p - 1) + g + (p + 4) &= 18 \\
2p + g + 3 &= 18 \\
g &= 15 - 2p
\end{align*}
\]

Since \(b = p - 1\) and \(b\) is a positive integer, the smallest positive integer value for \(p\) will be 2. Otherwise, \(b \leq 0\).

Since \(g = 15 - 2p\) and \(g\) is a positive integer, the largest positive integer value for \(p\) will be 7. Otherwise, \(g \leq 0\).

Therefore, the only possible values for \(p\) are \(\{2, 3, 4, 5, 6, 7\}\).

We will now look at each possible value of \(p\) and calculate the values of \(b, g\) and \(y\) in each case:

<table>
<thead>
<tr>
<th>Purple Card</th>
<th>Blue Card</th>
<th>Green Card</th>
<th>Yellow Card</th>
<th>Possible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(b = p - 1)</td>
<td>(g = 15 - 2p)</td>
<td>(y = p + 4)</td>
<td>Yes or No</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>11</td>
<td>6</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
<td>8</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>No ((p = g))</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>3</td>
<td>10</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>1</td>
<td>11</td>
<td>Yes, (p \neq b \neq g \neq y).</td>
</tr>
</tbody>
</table>

Therefore, there are 5 different possibilities for the numbers on the purple, blue, green and yellow cards, respectively.

They are \((p, b, g, y) = (2, 1, 11, 6), (3, 2, 9, 7), (4, 3, 7, 8), (6, 5, 3, 10), (7, 6, 1, 11)\).