Problem of the Week
Problem E and Solution
Parabolic Move

Problem
The parabola $y = -x^2 + 4$ has vertex labelled $P$ and intersects the $x$-axis at the points labelled $A$ and $B$. The parabola is translated from its original position so that its vertex moves along the line $y = x + 4$ to the point $Q$. In this new position, the parabola intersects the $x$-axis at the points labelled $B$ and $C$. Determine the coordinates of $C$.

Solution
For the original parabola $y = -x^2 + 4$, the vertex is $P(0, 4)$ and the $x$-intercepts are $A(-2, 0)$ and $B(2, 0)$.
Let the vertex of the translated parabola be $Q(q, p)$. Since the new parabola is a translation of the original, the equation of this new parabola is $y = -(x - q)^2 + p$.
Since $Q$ lies on the line $y = x + 4$, we have $p = q + 4$ and the equation of the new parabola is $y = -(x - q)^2 + q + 4$.
Since $B(2, 0)$ lies on the new parabola, we can substitute $(2, 0)$ into this equation:

\[
\begin{align*}
0 &= -(2-q)^2 + q + 4 \\
0 &= -(q^2 - 4q + 4) + q + 4 \\
0 &= -q^2 + 4q - 4 + q + 4 \\
0 &= -q^2 + 5q \\
0 &= -q(q-5)
\end{align*}
\]

Therefore, $q = 0$ or $q = 5$. The value $q = 0$ corresponds to point $P(0, 4)$ in the original parabola. Therefore, $q = 5$ and the vertex of the new parabola is $(5, 9)$ and the equation of this parabola is $y = -(x - 5)^2 + 9$.
Since $C$ is an $x$-intercept of this parabola, to determine $C$ set $y = 0$ in the equation for the parabola and solve for $x$:

\[
\begin{align*}
0 &= -(x - 5)^2 + 9 \\
(x - 5)^2 &= 9 \\
x - 5 &= \pm 3 \\
x &= 8, 2
\end{align*}
\]

The value $x = 2$ corresponds to point $B$, and the value $x = 8$ corresponds to point $C$. Therefore, the coordinates of $C$ are $(8, 0)$. 