Problem of the Week
Problem E
Up, Down, Up, Down, ...

The sequence \( \{1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \cdots \} \) is called an alternating sequence since the terms alternate positive and negative. In this particular sequence, the value of any term in an odd position in the sequence is the term number and the value of any term in an even position in the sequence is the term number multiplied by \(-1\).

Let \( S(n) \) be the sum of the first \( n \) terms of the sequence. Then it follows that

\[
\begin{align*}
S(1) &= 1 \\
S(3) &= 1 - 2 + 3 = 2 \\
S(5) &= 1 - 2 + 3 - 4 + 5 = 3 \\
S(7) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 = 4 \\
S(9) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 = 5 \\
S(2) &= 1 - 2 = -1 \\
S(4) &= 1 - 2 + 3 - 4 = -2 \\
S(6) &= 1 - 2 + 3 - 4 + 5 - 6 = -3 \\
S(8) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 = -4 \\
S(10) &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 = -5
\end{align*}
\]

The following are two unproven results from the above sums.

If \( n \) is an even positive integer, i.e., \( n = 2k \) where \( k \) is a positive integer, then

\[ S(n) = S(2k) = -k. \]

For example, \( S(6) = S(2 \times 3) = -3. \)

If \( n \) is an odd positive integer, i.e., \( n = 2m - 1 \) where \( m \) is a positive integer, then

\[ S(n) = S(2m - 1) = m. \]

For example, \( S(9) = S(2 \times 5 - 1) = 5. \)

Suppose that \( a \) and \( b \) are any positive integers. Using the two results above, show that

\[ S(a) + S(b) + S(a + b) = 1 \]

when both \( a \) and \( b \) are odd.

As an extension, show that \( S(a) + S(b) + S(a + b) = 1 \) only when both \( a \) and \( b \) are odd. As a further extension, prove that the two given results are true.