Problem of the Week

Problem D and Solution

Running Out of Digits

Problem

You initially have 100 of each digit from 0 to 9. This means you have 1000 digits in total. Now start counting by ones, from 1. Each time you say a number you must remove the digits required to make the number from your stockpile of digits. What is the largest number you can count to without running out of the digits needed to form the number?

Solution

In the integers 1 to 99, each digit, other than zero, will be used the same number of times. Zero will be used 9 less times than every other digit. As soon as you get to the number 100, the 1s will be used more frequently. So, let’s just count the number of 1s used until we run out.

From 1 to 99, a 1 is used in the units digit 10 times: \{1, 11, 21, \cdots, 91\}. There are 10 numbers with a tens digit of 1: \{10, 11, 12, \cdots, 18, 19\}. Therefore, by the time we reach 99, we have used \(10 + 10 = 20\) ones.

From 100 to 109, we use 10 ones plus another in the number 101. This makes a total of 11 more ones, so we have now used \(20 + 11 = 31\) ones.

From 110 to 119 we use 10 ones for hundreds digits and 10 ones for tens digits and one 1 for the units digit in 111. This makes a total of 21 more ones, so we have now used \(31 + 21 = 52\) ones.

For 120 to 129, 130 to 139, 140 to 149, and 150 to 159 we use the same number of ones as we did counting from 100 to 109. So, to get from 120 to 159 we use \(11 + 11 + 11 + 11 = 44\) ones, and have now used a total of \(52 + 44 = 96\) ones.

Now we can count until we use up the 4 remaining ones: 160, 161 and 162.

When we try to count 163, there are no ones remaining.

Therefore, the largest number we can count to before running out of the necessary digits to create the number is 162.