Problem

Timmy creates nine-digit positive integers by using the digits from 1 to 9 each exactly once. How many of these nine-digit integers are a multiple of 4 or have a units digit that is an 8?

Solution

In creating a 9-digit positive integer with distinct digits from 1 to 9, there are 9 choices for the first digit, 8 choices for the second digit, 7 choices for the third digit, and so on. So the number of 9-digit numbers with distinct digits is $9 \times 8 \times 7 \times \cdots \times 2 \times 1 = 9! = 362,880$.

First, we will count the number of 9-digit positive integers with distinct digits from 1 to 9 that end in an 8. If the 9-digit integer ends in 8, there is only one choice for the unit’s digit. There are then 8! ways to create the remaining 8-digit number. Thus, the number of 9-digit positive integers with distinct digits from 1 to 9 that end in 8 is equal to $1 \times 8! = 40,320$.

Next, we will count the number of 9-digit positive integers with distinct digits from 1 to 9 that are divisible by 4. For a number to be divisible by 4, the last two digits of the number must be divisible by 4. There are 24 positive integers less than 100 that are divisible by 4. This group includes two 1-digit numbers, 4 and 8, which must be excluded. It also includes four 2-digit numbers ending in zero, 20, 40, 60 and 80. Since zero is not one of the nine possible digits, these numbers must be excluded. And finally, the list includes two numbers, 44 and 88, which have repeated digits. Since each digit can be used only once, these numbers must be excluded. Therefore, there are $24 - 2 - 4 - 2 = 16$ valid 2-digit numbers which are divisible by 4. There are then 7! ways to create the remaining 7-digit number. Therefore, the number of 9-digit positive integers with distinct digits from 1 to 9 that are divisible by 4 is equal to $16 \times 7! = 80,640$.

If we added the $1 \times 8!$ and the $16 \times 7!$ we would be double counting the integers that end in 8 and are divisible by 4. So, we must subtract off the number of integers which are divisible by 4 and end in 8 since they have been counted twice. There are three 2-digit numbers with distinct digits that end in 8 and are divisible by 4. They are 28, 48 and 68. As before, there are 7! ways to create the remaining 7-digit number. So the number of 9-digit positive integers with distinct digits from 1 to 9 ending in 8 and divisible by 4 is equal to $3 \times 7! = 15,120$.

Therefore, the number of 9-digit integers that have distinct digits from 1 to 9 and are a multiple of 4 or have a units digit that is an 8 is equal to $40,320 + 80,640 - 15,120 = 105,840$. 