Problem of the Week
Problem E and Solution
Maybe One-Third?

Problem

In the diagram, square $OABC$ is positioned with $O$ at the origin $(0,0)$, $A$ on the positive $y$-axis, $C$ on the positive $x$-axis, and $B$ in the first quadrant. Side $OA$ is trisected by points $F$ and $G$ so that $OF = FG = GA = 100$. Side $OC$ is trisected by points $D$ and $E$ so that $OD = DE = EC = 100$. Line segment $BE$ intersects line segment $CF$ at $H$. If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?

Solution

Since $OF = 100$ and $F$ is on the positive $y$-axis, the coordinates of $F$ are $(0,100)$.

Since $OD = DE = 100$, it follows that $OE = 200$. Since $E$ is on the positive $x$-axis, the coordinates of $E$ are $(200,0)$.

Since $OD = DE = EC = 100$, it follows that the side length of the square is $OC = 300$. Since $C$ is on the positive $x$-axis, the coordinates of $C$ are $(300,0)$.

It then follows that the coordinates of $B$ are $(300,300)$.

The diagram has been updated to reflect the new information.

We will proceed to find the coordinates of $H$.

Find the equation of the line through $B(300,300)$ and $E(200,0)$.

The slope of $BE = \frac{300-0}{300-200} = 3$. We substitute $x = 200$, $y = 0$ and $m = 3$ into $y = mx + b$ to find $b$. Then $0 = 3(200) + b$ and $b = -600$ follows. The equation of the line through $BE$ is $y = 3x - 600$. (1)

Find the equation of the line through $C(300,0)$ and $F(0,100)$.

The slope of $CF = \frac{100-0}{0-300} = -\frac{1}{3}$. Since $F(0,100)$ is on the $y$-axis, the $y$-intercept is 100. It follows that the equation of the line through $C$ and $F$ is $y = -\frac{1}{3}x + 100$. (2)
Find the coordinates of $H$, the intersection of the two lines.

At the intersection, the $x$-coordinates are equal and the $y$-coordinates are equal. In (1) and (2), since $y = y$, then

$3x - 600 = -\frac{1}{3}x + 100 \implies 9x - 1800 = -x + 300 \implies 10x = 2100 \implies x = 210$

Substituting $x = 210$ into (1), $y = 3(210) - 600 = 30$. Therefore, the coordinates of $H$ are $(210, 30)$.

At this point we could follow one of two approaches. The first approach would be to find the area of the shaded regions indirectly, by first determining the area of the unshaded regions and then subtracting this from the area of the square. We will leave this approach to the solver.

Our second approach, which is below, is to calculate the areas of the shaded triangles directly.

The slope of $BE$ is 3 and the slope of $CF$ is $-\frac{1}{3}$. Since these slopes are negative reciprocals, we know that $BE \perp CF$. It follows that $\triangle BHF$ and $\triangle CHE$ are right-angled triangles.

We will now proceed with finding the side lengths necessary to calculate the area of each shaded triangle.

We first find the area of $\triangle BHF$.

$BH = \sqrt{(300 - 210)^2 + (300 - 30)^2}$
$= \sqrt{90^2 + 270^2}$
$= \sqrt{90^2(1 + 3^2)}$
$= 90\sqrt{10}$

$HF = \sqrt{(0 - 210)^2 + (100 - 30)^2}$
$= \sqrt{210^2 + 70^2}$
$= \sqrt{70^2(3^2 + 1)}$
$= 70\sqrt{10}$

$\text{Area } \triangle BHF = BH \times HF \div 2$
$= 90\sqrt{10} \times 70\sqrt{10} \div 2$
$= 31500$
Next we find the area of \( \triangle CHE \).

\[
CH = \sqrt{(300 - 210)^2 + (0 - 30)^2} = \sqrt{90^2 + 30^2} = \sqrt{30^2(3^2 + 1)} = 30\sqrt{10}
\]

\[
HE = \sqrt{(210 - 200)^2 + (30 - 0)^2} = \sqrt{10^2 + 30^2} = \sqrt{10^2(1 + 3^2)} = 10\sqrt{10}
\]

Area \( \triangle CHE \) = \( CH \times HE \div 2 \)

\[
= 30\sqrt{10} \times 10\sqrt{10} \div 2
\]

\[
= 1500
\]

We can now calculate the total area shaded, the area of square \( OABC \), and the fraction of the area of the square that is shaded.

Total Area Shaded = Area \( \triangle BHF \) + Area \( \triangle CHE \)

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= 31500 + 1500
\]

\[
= 33000
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Area \( OABC \) = \( OA \times OC \)

\[
= 300 \times 300
\]

\[
= 90000
\]

Fraction of Total Area Shaded = \( \frac{\text{Area } \triangle BHF}{\text{Area } OABC} \)

\[
= \frac{33000}{90000}
\]

\[
= \frac{11}{30}
\]

Therefore, \( \frac{11}{30} \) of the total area of the square is shaded. This, in fact, is more than one-third.