Problem of the Week
Problem D and Solution
That Triangle

Problem

In the diagram, $ABCD$ is a rectangle. Point $E$ is outside the rectangle so that $\triangle AED$ is an isosceles right-angled triangle with hypotenuse $AD$. Point $F$ is the midpoint of $AD$, and $EF$ is perpendicular to $AD$. If $BC = 4$ and $AB = 3$, determine the area of $\triangle EBD$.

Solution

Since $ABCD$ is a rectangle then $AD = BC = 4$.
Since $F$ is the midpoint of $AD$, then $AF = FD = 2$.
Since $\triangle AED$ is an isosceles right-angled triangle, then $\angle EAD = 45^\circ$.
Now in $\triangle EAF$, $\angle EAF = \angle EAD = 45^\circ$ and $\angle AFE = 90^\circ$.
Since the sum of the angles in a triangle is $180^\circ$, then $\angle AEF = 180^\circ - 90^\circ - 45^\circ = 45^\circ$. Therefore, $\triangle EAF$ has two equal angles and is therefore an isosceles right-angled triangle.
Therefore, $EF = AF = 2$.
From this point we are going to look at two different solutions.

Solution 1:

We calculate the area of $\triangle EBD$ by adding the areas of $\triangle BAD$ and $\triangle AED$ and subtracting the area of $\triangle ABE$.
Since $AB = 3$, $DA = 4$, and $\angle DAB = 90^\circ$, then the area of $\triangle BAD$ is $\frac{1}{2}(3)(4) = 6$.
Since $AD = 4$, $EF = 2$, and $EF$ is perpendicular $AD$, then the area of $\triangle AED$ is $\frac{1}{2}(4)(2) = 4$.
At the right, when we look at $\triangle ABE$ with the base being $AB$, then its height is the length of $AF$.
Therefore, the area of $\triangle ABE$ is $\frac{1}{2}(3)(2) = 3$.
Therefore, the area of $\triangle EBD$ is $6 + 4 - 3 = 7$. 
Solution 2:

Extend $BA$ to $G$ and $CD$ to $H$ so that $GH$ is perpendicular to each $GB$ and $HC$ and so that $GH$ passes through $E$.

Each of $GAFE$ and $EFDH$ has three right angles (at $G$, $A$, and $F$, and $F$, $D$, and $H$, respectively), so each of these is a rectangle.

Since $AF = EF = FD = 2$, then each of $GAFE$ and $EFDH$ is a square with side length 2.

Now $GBCH$ is a rectangle with $GB = 2 + 3 = 5$ and $BC = 4$.

The area of $\triangle EBD$ is equal to the area of rectangle $GBCH$ minus the areas of $\triangle EGB$, $\triangle BCD$, and $\triangle DHE$.

Rectangle $GBCH$ is 5 by 4, and so has area $5 \times 4 = 20$.

Since $EG = 2$ and $GB = 5$ and $EG$ is perpendicular to $GB$, then the area of $\triangle EGB$ is $\frac{1}{2}(EG)(GB) = \frac{1}{2}(2)(5) = 5$.

Since $BC = 4$ and $CD = 3$ and $BC$ perpendicular to $CD$, then the area of $\triangle BCD$ is $\frac{1}{2}(BC)(CD) = \frac{1}{2}(4)(3) = 6$.

Since $DH = HE = 2$ and $DH$ is perpendicular to $EH$, then the area of $\triangle DHE$ is $\frac{1}{2}(DH)(HE) = \frac{1}{2}(2)(2) = 2$.

Therefore, the area of $\triangle EBD$ is $20 - 5 - 6 - 2 = 7$. 