Problem of the Week
Problem C and Solution
Locate the Fourth Vertex

Problem

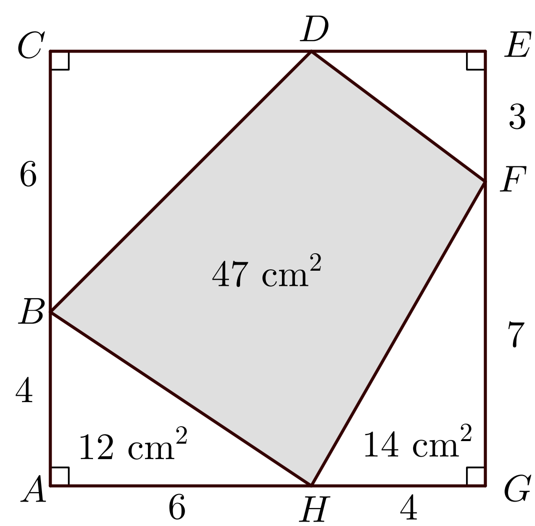
Quadrilateral $BDFH$ is constructed so that each vertex is on a different side of square $ACEG$. Vertex B is on side AC so that $AB = 4$ cm and $BC = 6$ cm. Vertex F is on EG so that $EF = 3$ cm and $FG = 7$ cm. Vertex H is on GA so that $GH = 4$ cm and $HA = 6$ cm. The area of quadrilateral $BDFH$ is 47 cm^2.

The fourth vertex of quadrilateral $BDFH$, labelled D, is located on side CE so that the lengths of CD and DE are both positive integers.

Determine the lengths of CD and DE.

Solution
Solution 1

Since $ACEG$ is a square and the length of $AG = AH + HG = 6 + 4 = 10$ cm, then the length of each side of the square is 10 cm and the area is $10 \times 10 = 100$ cm^2.



We can determine the area of triangles BAH and FGH using the formula area $= \frac{\text{base}\times\text{height}}{2}$.

In $\triangle BAH$, since BA is perpendicular to AH, we can use BA as the height and AH as the base. The area of $\triangle BAH$ is $\frac{6\times4}{2} = 12$ cm^2.

In $\triangle FGH$, since FG is perpendicular to GH, we can use FG as the height and GH as the base. The area of $\triangle FGH$ is $\frac{4\times7}{2} = 14$ cm^2.

The area of $\triangle BCD$ plus the area of $\triangle FED$ must be the total area minus the three known areas. That is,
Area $\triangle BCD$ + Area $\triangle FED = 100 - 12 - 47 - 14 = 27$ cm^2.

CD and DE are both positive integers and $CD + DE = 10$. We will systematically check all possible values for CD and DE to determine the values which produce the correct area.

CD	DE	Area $\triangle BCD$	Area $\triangle FED$	Area $\triangle BCD$ + Area $\triangle FED$
1	9	$1 \times 6 \div 2 = 3$	$9 \times 3 \div 2 = 13.5$	$3 + 13.5 = 16.5 \neq 27$
2	8	$2 \times 6 \div 2 = 6$	$8 \times 3 \div 2 = 12$	$6 + 12 = 18 \neq 27$
3	7	$3 \times 6 \div 2 = 9$	$7 \times 3 \div 2 = 10.5$	$9 + 10.5 = 19.5 \neq 27$
4	6	$4 \times 6 \div 2 = 12$	$6 \times 3 \div 2 = 9$	$12 + 9 = 21 \neq 27$
5	5	$5 \times 6 \div 2 = 15$	$5 \times 3 \div 2 = 7.5$	$15 + 7.5 = 22.5 \neq 27$
6	4	$6 \times 6 \div 2 = 18$	$4 \times 3 \div 2 = 6$	$18 + 6 = 24 \neq 27$
7	3	$7 \times 6 \div 2 = 21$	$3 \times 3 \div 2 = 4.5$	$21 + 4.5 = 25.5 \neq 27$
8	2	$8 \times 6 \div 2 = 24$	$2 \times 3 \div 2 = 3$	$24 + 3 = 27$
9	1	$9 \times 6 \div 2 = 27$	$1 \times 3 \div 2 = 1.5$	$27 + 1.5 = 28.5 \neq 27$

Therefore, when $CD = 8$ cm and $DE = 2$ cm, the area of quadrilateral $BDFH$ is 47 cm^2.

The second solution is more algebraic and will produce a solution for any lengths of CD and DE between 0 and 10 cm.

Solution 2

This solution begins the same as Solution 1. Algebra is introduced to complete the solution.

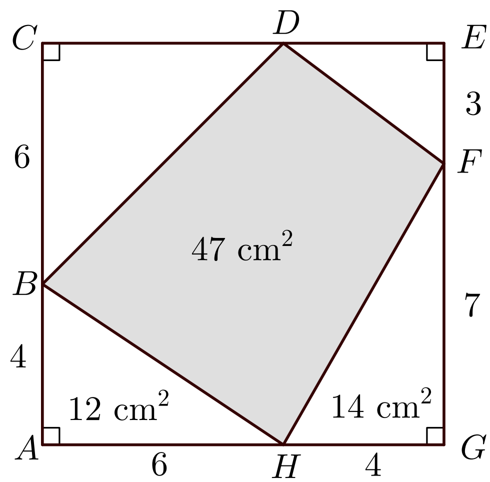
Since $ACEG$ is a square and the length of $AG = AH + HG = 6 + 4 = 10$ cm, then the length of each side of the square is 10 cm and the area is $10 \times 10 = 100$ cm^2.

We can determine the area of the triangles BAH and FGH using the formula $\frac{\text{base}\times\text{height}}{2}$.

In $\triangle BAH$, since BA is perpendicular to AH, we can use BA as the height and AH as the base. The area of $\triangle BAH$ is $\frac{6\times 4}{2} = 12$ cm^2.

In $\triangle FGH$, since FG is perpendicular to GH, we can use FG as the height and GH as the base. The area of $\triangle FGH$ is $\frac{4\times 7}{2} = 14$ cm^2.

The area of $\triangle BCD$ plus the area of $\triangle FED$ must be the total area minus the three known areas. That is, Area $\triangle BCD$ + Area $\triangle FED = 100 - 12 - 47 - 14 = 27$ cm^2.



Let the length of CD be n cm. Then the length of DE is $(10 - n)$ cm.

The area of $\triangle BCD$ is $\dfrac{BC \times CD}{2} = \dfrac{6 \times n}{2} = 3n$.

The area of $\triangle FED$ is $\dfrac{FE \times DE}{2} = \dfrac{3 \times (10-n)}{2} = \dfrac{10 - n + 10 - n + 10 - n}{2} = \dfrac{30 - 3n}{2}$.

Therefore,

$$\begin{aligned}
\text{Area } \triangle BCD + \text{Area } \triangle FED &= 27 \\
3n + \frac{30-3n}{2} &= 27 \\
\text{Multiplying both sides by 2:} \qquad 6n + 30 - 3n &= 54 \\
3n + 30 &= 54 \\
3n &= 24 \\
n &= 8
\end{aligned}$$

Therefore, the length of CD is 8 cm and the length of DE is 2 cm.

The algebra presented in Solution 2 may not be familiar to all students at this level.