Problem of the Week
Problem C and Solution
These Primes are Squares!

Problem

The number 7 has only two positive factors, 1 and itself. A positive integer greater than 1 whose only positive factors are 1 and itself is said to be *prime*.
A *perfect square* is an integer created by multiplying an integer by itself. The number 25 is a perfect square since it is 5×5 or 5^2.
Determine the smallest perfect square that has three different prime numbers as factors.

EXTENSION: Determine all perfect squares less than 10 000 that have three different prime numbers as factors.

Solution

The problem itself is not very difficult once you determine what it is asking. Let's begin by examining some perfect squares.

The numbers 4 and 9 are both perfect squares that have only one prime number as a factor, $4 = 2^2$ and $9 = 3^2$. The number 36 is a perfect square since $36 = 6^2$. However, the number $6 = 2 \times 3$ so $36 = (2 \times 3)^2 = 2 \times 3 \times 2 \times 3 = 2^2 \times 3^2$. So 36 is the product of the squares of two different prime numbers.

Notice that when we write a perfect square as a product of prime factors, each prime factor appears an even number of times in the product. This is because the perfect square is created by multiplying an integer by itself, so all of the prime factors of the integer appear twice.

To create the smallest perfect square with three different prime factors, we should choose the three smallest prime numbers, namely 2, 3, and 5. If we include any primes larger than these, then their product will be larger and so the perfect square will be larger. To make our number a perfect square, we should have each prime factor appear twice. This gives us $2^2 \times 3^2 \times 5^2 = 4 \times 9 \times 25 = 900$. It should be noted that $900 = 30^2 = (2 \times 3 \times 5)^2$.

Therefore, the smallest perfect square with three different prime numbers as factors is 900.

EXTENSION ANSWER:

Since perfect squares have an even number of each prime factor in their factorization, we can find all the perfect squares less than 10 000 with three different prime factors as numbers by systematically trying different combinations of prime numbers. The table below lists all the perfect squares less than 10 000 with three different prime factors.

$2^2 \times 3^2 \times 5^2 = 900$	$2^2 \times 2^2 \times 3^2 \times 5^2 = 3600$	$2^2 \times 3^2 \times 3^2 \times 5^2 = 8100$
$2^2 \times 3^2 \times 7^2 = 1764$	$2^2 \times 2^2 \times 3^2 \times 7^2 = 7056$	
$2^2 \times 3^2 \times 11^2 = 4356$		
$2^2 \times 3^2 \times 13^2 = 6084$		
$2^2 \times 5^2 \times 7^2 = 4900$		