Problem of the Week
Problem D and Solution
More Power, Mr. Scott!

Problem

Mr. Scott likes to pose interesting problems to his Mathematics classes. Today, he started with the expression $6^{2020} + 7^{2020}$. He stated that the expression was not equivalent to 13^{2020} and that he was not interested in the actual sum. His question to his class and to you is, "What are the final two digits of the sum?"

Solution

Solution 1

Let's start by examining the last two digits of various powers of 7.

$7^1 = \mathbf{07}$	$7^2 = \mathbf{49}$	$7^3 = \mathbf{343}$	$7^4 = 24\mathbf{01}$
$7^5 = 16\,8\mathbf{07}$	$7^6 = 117\,6\mathbf{49}$	$7^7 = 823\,5\mathbf{43}$	$7^8 = 5\,764\,8\mathbf{01}$

Notice that the last two digits repeat every four powers of 7. If the pattern continues, then 7^9 ends with 07, 7^{10} ends with 49, 7^{11} ends with 43, 7^{12} ends with 01, and so on. We can simply compute these powers of 7 to verify this for these examples, but let's justify why this pattern continues in general. If a power ends in "07", then the last 2 digits of the next power are the same as the last 2 digits of the product $07 \times 7 = 49$. That is, the last 2 digits of the next power are "49". If a power ends in "49", then the last 2 digits of the next power are the same as the last two digits of the product $49 \times 7 = 343$. That is, the last two digits of the next power are "43". If a power ends in "43", then the last 2 digits of the next power are the same as the last two digits of the product $43 \times 7 = 301$. That is, the last two digits of the next power are "01". Finally, if a power ends in "01", then the last 2 digits of the next power are the same as the last two digits of the product $01 \times 7 = 07$. That is, the last two digits of the next power are "07". Therefore, starting with the first power of 7, every four consecutive powers of 7 will have the last two digits 07, 49, 43, and 01.

We need to determine the number of complete cycles by dividing 2020 by 4. Since $2020 \div 4 = 505$, there are 505 complete cycles. This means that 7^{2020} is the last power of 7 in the 505^{th} cycle and therefore ends with 01.

Next we will examine the last two digits of various powers of 6.

$6^1 = \mathbf{06}$	$6^2 = \mathbf{36}$	$6^3 = 2\mathbf{16}$	$6^4 = 12\mathbf{96}$	$6^5 = 77\mathbf{76}$	$6^6 = 46\,6\mathbf{56}$
	$6^7 = 279\,9\mathbf{36}$	$6^8 = 1\,679\,6\mathbf{16}$	$6^9 = 10\,077\,6\mathbf{96}$	$6^{10} = 60\,466\,1\mathbf{76}$	$6^{11} = 362\,797\,0\mathbf{56}$

Notice that the last two digits repeat every five powers of 6 starting with the 2^{nd} power of 6. This pattern can be justified using an argument similar to the one above for powers of 7. So 6^{12} ends with 36, 6^{13} ends with 16, 6^{14} ends with 96, 6^{15} ends with 76, 6^{16} ends with 56, and so on. Starting with the second power of 6, every five consecutive powers of 6 will have the last two digits 36, 16, 96, 76, and 56.

We need to determine the number of complete cycles in 2020 by first subtracting 1 to allow for 06 at the beginning of the list and then dividing $2020 - 1$ or 2019 by 5. Since $2019 \div 5 = 403$ remainder 4, there are 403 complete cycles and $\frac{4}{5}$ of another cycle. Since $403 \times 5 = 2015$, $6^{2015+1} = 6^{2016}$ is the last power of 6 in the 403^{rd} cycle and therefore ends with 56.

To go $\frac{4}{5}$ of the way into the next cycle tells us that the number 6^{2020} ends with the fourth number in the pattern, namely 76. In fact, we know that 6^{2017} ends with 36, 6^{2018} ends with 16, 6^{2019} ends with 96, 6^{2020} ends with 76, and 6^{2021} ends with 56 because they would be the numbers in the 404^{th} complete cycle.

Therefore, 6^{2020} ends with the digits 76.

The final two digits of the sum $6^{2020} + 7^{2020}$ are found by adding the final two digits of 6^{2020} and 7^{2020}. Therefore, the final two digits of the sum are $01 + 76 = 77$.

Solution 2

From the first solution, we saw that the last two digits of powers of 7 repeat every 4 consecutive powers. We also saw that the last two digits of powers of 6 repeat every 5 consecutive powers after the first power of 6.

Let's start at the second powers of both 7 and 6. We know that the last two digits of 7^2 are 49 and the last two digits of 6^2 are 36. When will this combination of last two digits occur again? The cycle length for powers of 7 is 4 and the cycle length for powers of 6 is 5.

The least common multiple of 4 and 5 is 20. It follows that 20 powers after the second power, the last two digits of the powers of 7 and 6 will end with the same two digits as the second powers of each. That is, the last two digits of 7^{22} and 7^2 are the same, namely 49. And, the last two digits of 6^{22} and 6^2 are the same, namely 36. The following table illustrates this repetition.

Powers	7^2	7^3	7^4	7^5	7^6	7^7	7^8	7^9	7^{10}	7^{11}	7^{12}	7^{13}	7^{14}	7^{15}	7^{16}	7^{17}	7^{18}	7^{19}	7^{20}	7^{21}	7^{22}
Last 2 digits	**49**	43	01	07	49	43	01	07	49	43	01	07	49	43	01	07	49	43	01	07	**49**

Powers	6^2	6^3	6^4	6^5	6^6	6^7	6^8	6^9	6^{10}	6^{11}	6^{12}	6^{13}	6^{14}	6^{15}	6^{16}	6^{17}	6^{18}	6^{19}	6^{20}	6^{21}	6^{22}
Last 2 digits	**36**	16	96	76	56	36	16	96	76	56	36	16	96	76	56	36	16	96	76	56	**36**

Since 2000 is a multiple of 20, we then know that the 2022^{nd} power of 7 will end with 49 and that the 2022^{nd} power of 6 will end in 36.

Working backwards through the cycle of the last two digits of powers of 7, it follows that the 2021^{st} power of 7 ends in 07 and that the 2020^{th} power of 7 ends in 01.

Working backwards through the cycle of the last two digits of powers of 6, it follows that the 2021^{st} power of 6 ends in 56 and that the 2020^{th} power of 6 ends in 76.

The final two digits of the sum $6^{2020} + 7^{2020}$ are found by adding the final two digits of 6^{2020} and 7^{2020}. Therefore, the final two digits of the sum are $01 + 76 = 77$.