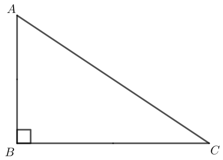


Problem of the Week

Problem E and Solution

Medians

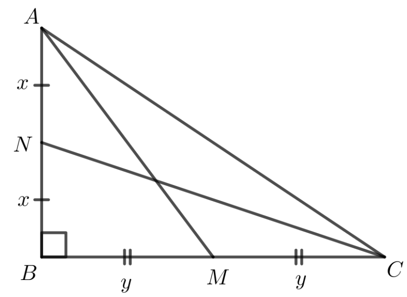
Problem

In $\triangle ABC$, $\angle ABC = 90^\circ$. A median is drawn from A to side BC, meeting BC at M such that $AM = 5$. A second median is drawn from C to side AB, meeting AB at N such that $CN = 2\sqrt{10}$.

Determine the length of the longest side of $\triangle ABC$.

NOTE: In a triangle, a *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.

Solution



Since AM is a median, M is the midpoint of BC. Then $BM = MC$. Let $BM = MC = y$.
Since CN is a median, N is the midpoint of AB. Then $AN = NB$. Let $AN = NB = x$.

Since $\angle B = 90^\circ$, $\triangle NBC$ is a right-angled triangle. Using the Pythagorean Theorem,

$$\begin{aligned} NB^2 + BC^2 &= CN^2 \\ x^2 + (2y)^2 &= (2\sqrt{10})^2 \\ x^2 + 4y^2 &= 40 \qquad (1) \end{aligned}$$

Since $\angle B = 90^\circ$, $\triangle ABM$ is a right-angled triangle. Using the Pythagorean Theorem,

$$\begin{aligned} AB^2 + BM^2 &= AM^2 \\ (2x)^2 + y^2 &= 5^2 \\ 4x^2 + y^2 &= 25 \qquad (2) \end{aligned}$$

Adding equations (1) and (2), we get $5x^2 + 5y^2 = 65$ or $x^2 + y^2 = 13$.

The longest side of $\triangle ABC$ is the hypotenuse AC. Using the Pythagorean Theorem,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= (2x)^2 + (2y)^2 \\ &= 4x^2 + 4y^2 \\ &= 4(x^2 + y^2) \end{aligned}$$

Since $x^2 + y^2 = 13$, we have $AC^2 = 4(13)$. And since $AC > 0$, $AC = 2\sqrt{13}$ follows.

Therefore, the length of the longest side of $\triangle ABC$ is $2\sqrt{13}$.

NOTE: The solver could have instead solved a system of equations to find $x = 2$ and $y = 3$, and then proceed to solve for the longest side. The above approach was provided to expose the solver to alternate way to think about the solution of this problem.