



Problem of the Week

Problem E and Solution

Taking Grapes

Problem

Quinn and Birgitta are playing a game using a bowl of grapes. The rules of the game are as follows:

- Write the numbers 1, 2, 3, 4, and 5 on a whiteboard.
- Players take turns choosing a number from the whiteboard, removing that number of grapes from the bowl, and then erasing that number from the whiteboard.
- The game continues until all the numbers are erased or a player is not able to take any of the remaining numbers of grapes from the bowl.
- The last player who erased a number from the whiteboard is the winner.

The game starts with 8 grapes and Quinn goes first. Quinn has developed a winning strategy so that she is guaranteed to win this game, regardless of how many grapes Birgitta takes on her turns. Find all the possible first moves in Quinn's winning strategy. Justify your answer.

Solution

Quinn has five options to begin with. She can remove 1, 2, 3, 4, or 5 grapes. Let's look at each case.

- *Case 1:* Quinn removes 5 grapes on her first turn.

If Quinn removes 5 grapes, then Birgitta can remove the remaining 3 grapes and win the game. Therefore, Quinn could lose if she starts by removing 5 grapes.

- *Case 2:* Quinn removes 4 grapes on her first turn.

If Quinn removes 4 grapes, then there will be 4 grapes remaining. Birgitta can then remove 1, 2, or 3 grapes since Quinn will have erased the number 4 from the whiteboard.

If Birgitta removes 1 grape, then there will be 3 grapes remaining. Quinn can then remove the remaining 3 grapes and win the game.

If Birgitta removes 2 grapes, then there will be 2 grapes remaining. Quinn's only option will be to remove 1 grape, since the number 2 will have already been erased from the whiteboard. Then there will be 1 grape remaining, but Birgitta will not be able to remove it, since the number 1 will have been erased from the whiteboard on the previous turn. Therefore, Quinn will win the game.

If Birgitta removes 3 grapes, then there will be 1 grape remaining. Quinn can then remove the remaining grape and win the game.

Therefore, no matter what Birgitta does on her turn, Quinn can win if she starts the game by removing 4 grapes.



- *Case 3:* Quinn removes 3 grapes on her first turn.

If Quinn removes 3 grapes, then Birgitta can remove the remaining 5 grapes and win the game. Therefore, Quinn could lose if she starts by removing 3 grapes.

- *Case 4:* Quinn removes 2 grapes on her first turn.

If Quinn removes 2 grapes, then there will be 6 grapes remaining. Birgitta can then remove 1, 3, 4, or 5 grapes since Quinn will have erased the number 2 from the whiteboard.

If Birgitta removes 1 grape, then there will be 5 grapes remaining. Quinn can then remove the 5 remaining grapes and win the game.

If Birgitta removes 3 grapes, then there will be 3 grapes remaining. Quinn's only option will be to remove 1 grape, since the numbers 2 and 3 will have already been erased from the whiteboard. Then there will be 2 grapes remaining, but Birgitta will not be able to remove any of them, since the numbers 1 and 2 will have already been erased from the whiteboard. Therefore, Quinn will win the game.

If Birgitta removes 4 grapes, then there will be 2 grapes remaining. Quinn's only option will be to remove 1 grape, since the number 2 will have already been erased from the whiteboard. Then there will be 1 grape remaining, but Birgitta will not be able to remove it since the number 1 will have been erased from the whiteboard on the previous turn. Therefore, Quinn will win the game.

If Birgitta removes 5 grapes, then there will be 1 grape remaining. Quinn can then remove the remaining grape and win the game.

Therefore, no matter what Birgitta does on her turn, Quinn can win if she starts the game by removing 2 grapes.

- *Case 5:* Quinn removes 1 grape on her first turn.

If Quinn removes 1 grape, then there will be 7 grapes remaining. Birgitta can then remove 2, 3, 4, or 5 grapes.

If Birgitta removes 2 grapes, then there will be 5 grapes remaining. Quinn can remove the remaining 5 grapes and win the game.

If Birgitta removes 3 grapes, then there will be 4 grapes remaining. Quinn can remove the remaining 4 grapes and win the game.

If Birgitta removes 4 grapes, then there will be 3 grapes remaining. Quinn can remove the remaining 3 grapes and win the game.

If Birgitta removes 5 grapes, then there will be 2 grapes remaining. Quinn can remove the remaining 2 grapes and win the game.

Therefore, no matter what Birgitta does on her turn, Quinn can win if she starts the game by removing 1 grape.

Therefore, Quinn's winning strategy starts by removing 1, 2, or 4 grapes.