Problem of the Week
Problem E and Solution
Fold Once

Problem
A rectangular piece of paper, PQRS, has PQ = 30 cm and PS = 40 cm. The paper has grey lines on one side and is plain white on the other. The paper is folded so that the two diagonally opposite corners P and R coincide. This creates a crease along line segment AC, with A on PS and C on QR. Determine the length of AC.

Solution
Since PQRS is a rectangle, all angles inside PQRS are 90°. After the fold, R coincides with P. Label the point that S folds to as D. The angle at D is the same as the angle at S. Since PQRS is a rectangle, ∠PSR = 90° and SR = 30, and it follows that ∠PDA = 90° and PD = 30.

Let a represent the length of QC and b represent the length of AS. Then CR = QR − QC = 40 − a and PA = PS − AS = 40 − b.

Since S folds to D, it follows that AD = AS = b.

Since R folds to P, it follows that PC = CR = 40 − a.

All of the information is recorded on the following diagram.

Since ΔPQC is a right-angled triangle, we can use the Pythagorean Theorem to find a.

\[ QC^2 + PQ^2 = PC^2 \]
\[ a^2 + 30^2 = (40 − a)^2 \]
\[ a^2 + 900 = 1600 − 80a + a^2 \]
\[ 80a = 700 \]
\[ a = \frac{35}{4} \]
Since \( \triangle PDA \) is a right-angled triangle, we can use the Pythagorean Theorem to find \( b \).

\[
AD^2 + PD^2 = PA^2 \\
b^2 + 30^2 = (40 - b)^2 \\
b^2 + 900 = 1600 - 80b + b^2 \\
80b = 700 \\
b = \frac{35}{4}
\]

Therefore, \( a = b = \frac{35}{4} \) cm.

We still need to find the length of the crease. From \( A \) drop a perpendicular to \( QR \) intersecting \( QR \) at \( B \). Then \( ABRS \) is a rectangle. It follows that \( AB = SR = 30 \) and \( BR = AS = b \).

Also, \( CB = QR - QC - BR = 40 - a - b = 40 - \frac{35}{4} - \frac{35}{4} = \frac{90}{4} = \frac{45}{2} \) cm.

Using the Pythagorean Theorem in \( \triangle CAB \),

\[
AC^2 = AB^2 + CB^2 \\
= 30^2 + \left( \frac{45}{2} \right)^2 \\
= 900 + \frac{2025}{4} \\
= \frac{5625}{4}
\]

Since \( AC > 0 \), it follows that \( AC = \frac{75}{2} = 37.5 \) cm. Therefore, the length of the crease is \( \frac{75}{2} \) cm.