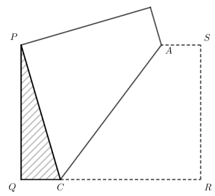


Problem of the Week

Problem E and Solution

Fold Once

Problem

A rectangular piece of paper, $PQRS$, has $PQ = 30$ cm and $PS = 40$ cm. The paper has grey lines on one side and is plain white on the other. The paper is folded so that the two diagonally opposite corners P and R coincide. This creates a crease along line segment AC, with A on PS and C on QR. Determine the length of AC.

Solution

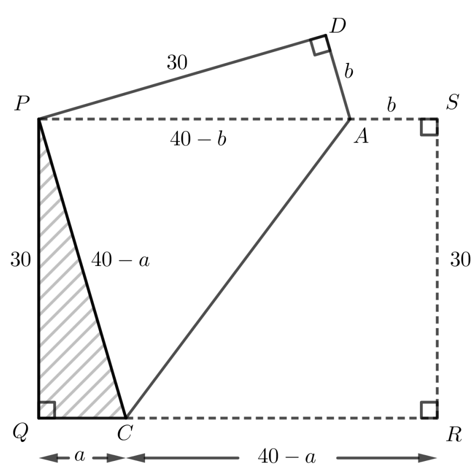
Since $PQRS$ is a rectangle, all angles inside $PQRS$ are 90°. After the fold, R coincides with P. Label the point that S folds to as D. The angle at D is the same as the angle at S. Since $PQRS$ is a rectangle, $\angle PSR = 90°$ and $SR = 30$, and it follows that $\angle PDA = 90°$ and $PD = 30$.

Let a represent the length of QC and b represent the length of AS.
Then $CR = QR - QC = 40 - a$ and $PA = PS - AS = 40 - b$.

Since S folds to D, it follows that $AD = AS = b$.

Since R folds to P, it follows that $PC = CR = 40 - a$.

All of the information is recorded on the following diagram.



Since $\triangle PQC$ is a right-angled triangle, we can use the Pythagorean Theorem to find a.

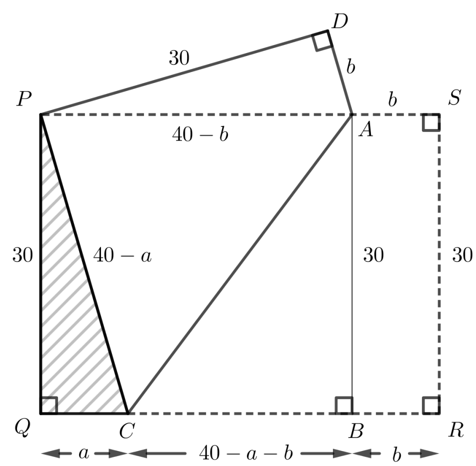
$$\begin{aligned} QC^2 + PQ^2 &= PC^2 \\ a^2 + 30^2 &= (40 - a)^2 \\ a^2 + 900 &= 1600 - 80a + a^2 \\ 80a &= 700 \\ a &= \frac{35}{4} \end{aligned}$$

Since $\triangle PDA$ is a right-angled triangle, we can use the Pythagorean Theorem to find b.

$$\begin{aligned} AD^2 + PD^2 &= PA^2 \\ b^2 + 30^2 &= (40-b)^2 \\ b^2 + 900 &= 1600 - 80b + b^2 \\ 80b &= 700 \\ b &= \frac{35}{4} \end{aligned}$$

Therefore, $a = b = \dfrac{35}{4}$ cm.

We still need to find the length of the crease. From A drop a perpendicular to QR intersecting QR at B. Then $ABRS$ is a rectangle. It follows that $AB = SR = 30$ and $BR = AS = b$. Also, $CB = QR - QC - BR = 40 - a - b = 40 - \dfrac{35}{4} - \dfrac{35}{4} = \dfrac{90}{4} = \dfrac{45}{2}$ cm.



Using the Pythagorean Theorem in $\triangle CAB$,

$$\begin{aligned} AC^2 &= AB^2 + CB^2 \\ &= 30^2 + \left(\frac{45}{2}\right)^2 \\ &= 900 + \frac{2025}{4} \\ &= \frac{5625}{4} \end{aligned}$$

Since $AC > 0$, it follows that $AC = \dfrac{75}{2} = 37.5$ cm. Therefore, the length of the crease is $\dfrac{75}{2}$ cm.