Problem of the Week
Problem E and Solution
Now I Know My ABCs

Problem
In triangle $ABC$, point $P$ lies on $AB$, point $Q$ lies on $BC$, and point $R$ lies on $AC$ such that $AQ$, $BR$, and $CP$ are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.

Determine the measure, in degrees, of $\angle ABC$, and the lengths, in centimetres, of $AB$, $BC$, and $CA$.

Note the diagram is not drawn to scale.

Solution
Let $BC = a$, $AC = b$, and $AB = c$. We will present two methods for determining that $21a = 24b = 56c$, and then continue on with the rest of the solution.

- **Method 1: Use Areas**
  We can find the area of $\triangle ABC$ by multiplying the length of the altitude (the height) by the corresponding base and dividing by 2. Therefore,
  \[
  \frac{AQ \times BC}{2} = \frac{BR \times AC}{2} = \frac{CP \times AB}{2}
  \]
  Substituting $AQ = 21$, $BR = 24$, and $CP = 56$, and multiplying through by 2 gives us $21a = 24b = 56c$.

- **Method 2: Use Trigonometry**
  In right-angled $\triangle ARB$, $\sin A = \frac{BR}{AB} = \frac{24}{c}$. In $\triangle APC$, $\sin A = \frac{CP}{AC} = \frac{56}{b}$. Putting these together gives $\frac{24}{c} = \frac{56}{b}$, or $24b = 56c$.
  In right-angled $\triangle BQA$, $\sin B = \frac{AQ}{AB} = \frac{21}{c}$. In $\triangle BPC$, $\sin B = \frac{CP}{BC} = \frac{56}{a}$. Putting these together gives $\frac{21}{c} = \frac{56}{a}$, or $21a = 56c$.
  Combining these gives $21a = 24b = 56c$. 
We now will continue on with the rest of the solution.

From $21a = 24b$ we obtain $b = \frac{21}{24}a = \frac{7}{8}a$, and from $21a = 56c$ we obtain $c = \frac{21}{56}a = \frac{3}{8}a$.

The ratio of the sides in $\triangle ABC$ is therefore $a : b : c = a : \frac{7}{8}a : \frac{3}{8}a = 8 : 7 : 3$. Let $BC = 8x$, $AC = 7x$, and $AB = 3x$, where $x > 0$.

Using the cosine law,

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos(\angle ABC)$$
$$\quad = (7x)^2 + (8x)^2 - 2(3x)(8x)\cos(\angle ABC)$$
$$\quad = 49x^2 + 64x^2 - 48x^2\cos(\angle ABC)$$
$$\quad = 73x^2 - 48x^2\cos(\angle ABC)$$

Since $x > 0$, we know $x^2 \neq 0$. So dividing by $x^2$,

$$49 = 73 - 48\cos(\angle ABC)$$

Rearranging,

$$48\cos(\angle ABC) = 24$$
$$\cos(\angle ABC) = \frac{1}{2}$$

Therefore, since $\triangle ABC$ is an acute triangle, we know $\angle ABC = 60^\circ$.

In right $\triangle BPC$,

$$\frac{CP}{BC} = \sin 60^\circ$$
$$BC = \frac{CP}{\sin 60^\circ}$$
$$BC = \frac{56}{\frac{\sqrt{3}}{2}}$$
$$BC = \frac{112}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$BC = \frac{112\sqrt{3}}{3}$$

However, $BC = 8x$. Therefore,

$$8x = \frac{112\sqrt{3}}{3}$$
$$x = \frac{14\sqrt{3}}{3}$$
$$3x = 14\sqrt{3}$$
$$7x = \frac{98\sqrt{3}}{3}$$

Therefore, $\angle ABC = 60^\circ$, and the side lengths of $\triangle ABC$ are $AB = 3x = 14\sqrt{3}$ cm, $AC = 7x = \frac{98\sqrt{3}}{3}$ cm, and $BC = \frac{112\sqrt{3}}{3}$ cm.