

Problem of the Week

Problem E and Solution

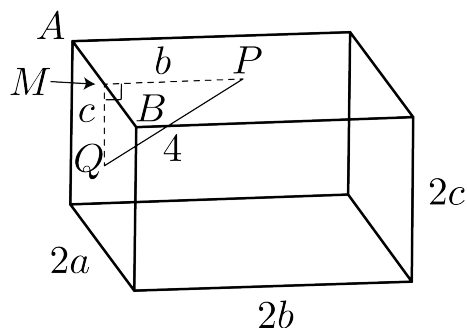
I Want More Volume

Problem

A rectangular prism is placed on a table. Points P , Q , and R lie on three different faces of the prism with P on the top face and Q and R on two adjacent side faces. Each point is located where the diagonals of the particular face intersect. Connecting these three points gives us $\triangle PQR$. If $PQ = 4$ cm, $QR = 5$ cm, and $RP = 6$ cm, determine the volume of the rectangular prism.

Solution

First, we label the top edge of the face containing point Q as AB and its midpoint as M . Let $AM = a$, $PM = b$, and $MQ = c$. Since the centre of a rectangle is where its diagonals intersect, P and Q are at the centres of their respective faces. Further, since the centre of a rectangle is also where the perpendicular bisectors of the sides of the rectangle meet, the rectangular prism has dimensions $2a$, $2b$, and $2c$, as shown in the following diagram.



Since $\angle PMQ = 90^\circ$, it follows that

$$b^2 + c^2 = 16 \quad (1)$$

Similarly, we can conclude the following.

$$a^2 + c^2 = 36 \quad (2)$$

$$a^2 + b^2 = 25 \quad (3)$$

Adding equations (1), (2), and (3) gives us the following.

$$2a^2 + 2b^2 + 2c^2 = 77$$

$$a^2 + b^2 + c^2 = \frac{77}{2} \quad (4)$$



Now, we subtract each of equations (1), (2), and (3) from equation (4) to obtain

$$a^2 = \frac{45}{2}, \quad b^2 = \frac{5}{2}, \quad \text{and} \quad c^2 = \frac{27}{2}$$

Multiplying a^2 , b^2 , and c^2 together gives the product

$$a^2b^2c^2 = \frac{45}{2} \times \frac{5}{2} \times \frac{27}{2} = \frac{6075}{8}$$

Then, taking the positive square root,

$$abc = \sqrt{\frac{6075}{8}} = \frac{45\sqrt{3}}{2\sqrt{2}} = \frac{45\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{45\sqrt{6}}{4}$$

To determine the volume of the rectangular prism, we multiply the side lengths together.

$$\begin{aligned} V &= (2a)(2b)(2c) \\ &= 8abc \\ &= 8 \left(\frac{45\sqrt{6}}{4} \right) \\ &= 90\sqrt{6} \text{ cm}^3 \end{aligned}$$

Therefore, the volume of the rectangular prism is $90\sqrt{6} \text{ cm}^3$.