$1 \times 2 = 3\,?$

Problem of the Week
Problem E and Solution
Three Lists

Problem

Ameya has two lists, List 1 and List 2, which each have six entries that are consecutive positive integers. The smallest entry in List 1 is a and the smallest entry in List 2 is b, and $a < b$.

Ameya creates a third list, List 3. The thirty-six entries in List 3 come from the product of each number in List 1 with each number of List 2. (There could be repeated numbers in List 3.)

Suppose that List 3 has 49 as an entry, has no entry that is multiple of 64, and has an entry larger than 75. Determine all possible pairs (a, b).

Solution

We will start by considering what the first condition tells us about the values of a and b, as it seems to be the most restrictive of the three conditions.

The first condition tells us that 49 must be the product of an integer from List 1 and an integer from List 2. Since $49 = 7^2$, 7 is prime, and all integers in the two lists are positive, these integers must be either 1 and 49, or 7 and 7.

Note: It is not possible for 49 to be obtained in both of these ways at once because if a list contains 49, then it cannot also contain 7. However, knowing this will not be important for our solution.

We will find all possible values of a and b by considering the two cases separately:

- Case 1: 49 was obtained in the third list by multiplying 1 and 49.

  Since the number 1 is in one of the lists, we must have either $a = 1$ or $b = 1$. The condition of $a < b$ means we must have $a = 1$. This means that List 1 must be

  $$1, 2, 3, 4, 5, 6$$

  and the number 49 must appear somewhere in List 2.

  Therefore, List 2 is one of the following six lists:

  $$44, 45, 46, 47, 48, 49$$
  $$45, 46, 47, 48, 49, 50$$
  $$46, 47, 48, 49, 50, 51$$
  $$47, 48, 49, 50, 51, 52$$
  $$48, 49, 50, 51, 52, 53$$
  $$49, 50, 51, 52, 53, 54$$

  Notice that $4 \times 48 = 192 = 64 \times 3$. Since 4 is in List 1, and no number in the third list can be a multiple of 64, then List 2 cannot contain the number 48. This leaves just one possibility for List 2:

  $$49, 50, 51, 52, 53, 54$$

This case gives exactly one possibility for the pair (a, b), namely $(1, 49)$.

We can verify that the third list for the pair $(a, b) = (1, 49)$ actually satisfies the second and third conditions. For the second condition, we note that $64 = 2^6$ and that we can get at most two factors of 2 from a number in List 1 and at most two factors of 2 from a number in List 2. It follows that any product in the third list will have at most 4 factors of 2, and hence cannot be a multiple of 64. For the third condition, we note that $2 \times 49 = 98$ is in the third list and is greater than 75.

- Case 2: 49 was obtained in the third list by multiplying 7 and 7.

  In this case, we know that the number 7 must appear in both List 1 and List 2. In order for this to happen we need to have $2 \leq a \leq 7$ and $2 \leq b \leq 7$. Since $a < b$, we actually must have $3 \leq b \leq 7$. (The smallest a can be is 2 and so b must be at least one more than that.)

  Since $3 \leq b \leq 7$, List 2 *must* contain the number 8. This means that to satisfy the second condition, List 1 *cannot* contain the number 8. Therefore, we must have $a = 2$. This means that List 1 must be

$$2, 3, 4, 5, 6, 7$$

  Since $7 \times 10 = 70$ and $7 \times 11 = 77$, the third list can only satisfy the third condition if List 2 contains a number at least as large as 11. This means we cannot have $b = 3$, $b = 4$, or $b = 5$, leaving the only possible values to be $b = 6$ or $b = 7$. These values produce the following possibilities for List 2:

$$6, 7, 8, 9, 10, 11$$

  or

$$7, 8, 9, 10, 11, 12$$

  Therefore, this case gives two additional possibilities for the pair (a, b), namely $(2, 6)$ and $(2, 7)$.

  We can verify that the third list for each of the the pairs $(a, b) = (2, 6)$ and $(a, b) = (2, 7)$ satisfies the second and third conditions using an argument similar to the one given in Case 1.

Combining the two cases, we conclude that there are exactly three pairs, (a, b), that satisfy all three conditions. They are $(1, 49)$, $(2, 6)$, and $(2, 7)$.