Problem of the Week
Problem E and Solution
A Very Large Prime

Problem

A *prime number* is an integer greater than 1 that has only two positive divisors: 1 and itself.

For some six-digit positive integer $216\,09d$ with ones (units) digit d, $2^{21609d} - 1$ is a very large prime number. In fact, the number contains 65 050 digits. The number begins with 746 093 103 064 661 343 and ends with the digit 7.

Determine the value of d.

Here are some facts which may be helpful when solving this problem:

1. If n is a positive integer and divisible by 3, then $2^n - 1$ is divisible by 7.
2. If n is a positive integer and divisible by 5, then $2^n - 1$ is divisible by 31.

Solution

To start, we will look for a pattern in the ones digit of powers of 2.

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64,\ 2^7 = 128,\ 2^8 = 256$$

It appears that the ones digit of powers of 2 repeat in the cycle $2,\ 4,\ 8,\ 6$. The next four powers of 2, $2^9,\ 2^{10},\ 2^{11}$, and 2^{12}, end with ones digits $2,\ 4,\ 8$, and 6, respectively, as expected. It turns out that this pattern continues, can you convince yourself of this?

Since $2^{21609d} - 1$ has ones digit 7, we will be interested in finding powers of 2 with ones digit 8.

From the pattern, we know that 2^{216088} has ones digit 6 since 216 088 is divisible by 4. It then follows that 2^{216089} has ones digit 2, 2^{216090} has ones digit 4, and 2^{216091} has ones digit 8. Since 2^{216091} has ones digit 8, it follows that 2^{216095} and 2^{216099} also each have ones digits 8.

Then $2^{216091} - 1$, $2^{216095} - 1$ and $2^{216099} - 1$ each have ones digit 7, and the only possible values of d are $1,\ 5$, and 9.

- If $d = 5$, then our six-digit number 216 095 is divisible by 5. From the facts given in the problem, it follows that $2^{216095} - 1$ is divisible by 31 and is therefore not a prime number. Thus, $d \neq 5$.

- If $d = 9$, then the sum of the digits of our six-digit positive integer 216 099 is 27. If the sum of the digits of an integer is divisible by 3, then the integer itself is divisible by 3. Since 27 is divisible by 3, it follows that 216 099 is divisible by 3. From the facts given in the problem, it follows that $2^{216099} - 1$ is divisible by 7 and is therefore not a prime number. Thus $d \neq 9$.

- It follows that we must have $d = 1$, and thus $2^{216091} - 1$ is a prime number ending in 7. In fact, this prime number is from a group of prime numbers called *Mersenne primes*. This number is the 31^{st} Mersenne prime and it was discovered in September of 1985. For more on Mersenne Primes, check out the Great Internet Mersenne Prime Search (GIMPS) at www.mersenne.org. According to GIMPS, as of March 2022, 51 Mersenne Primes are known. Perhaps you will be part of a team that will discover the next Mersenne Prime. There are prizes awarded when new discoveries are found and verified.

Extension: Can you prove the two facts given in the problem? To do so, you may need to look up the proof technique called "Proof by Mathematical Induction".