A	5	B
C	D	17
11	13	E

Problem of the Week
Problem C and Solution
Odd Boxes

Problem

The first 9 positive odd integers are placed in the 3 by 3 grid in such a way that the sum of the numbers in each row, column and main diagonal is the same. Four of the numbers are shown and the other five numbers are represented by the letters A, B, C, D, and E.

Determine the values of A, B, C, D, and E.

Solution

The final answer is $A = 15$, $B = 7$, $C = 1$, $D = 9$, and $E = 3$, which we will justify below in two different ways.

Solution 1

The numbers to be placed in the grid are 1, 3, 5, 7, 9, 11, 13, 15, and 17, the first 9 positive odd integers. Therefore, the sum of all of the numbers in the grid is $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81$. It follows that the sum of the sums of the three rows is 81. But each row has the same sum, so the sum of the numbers in each row is $81 \div 3 = 27$. We know that the sum of the numbers in each row, column and diagonal is the same. Therefore, the sum of the numbers in each column is also equal to 27 and the sum of the numbers in each diagonal is also equal to 27.

We can now use this information to determine the values in each cell of the 3 by 3 grid. In the third row, we know that $11 + 13 + E = 27$ or $24 + E = 27$ and $E = 3$ follows. Here is the updated grid:

A	5	B
C	D	17
11	13	3

In the second column, we know that $5 + D + 13 = 27$ and $D = 9$ follows. Here is the updated grid:

A	5	B
C	9	17
11	13	3

In second row, we know that $C + 9 + 17 = 27$ and $C = 1$ follows. Here is the updated grid:

A	5	B
1	9	17
11	13	3

In the first column, we know that $A + 1 + 11 = 27$ and $A = 15$ follows. Here is the updated grid:

15	5	B
1	9	17
11	13	3

In the first row, we know that $15 + 5 + B = 27$ and $B = 7$ follows. Here is the updated grid:

15	5	7
1	9	17
11	13	3

Therefore, $A = 15$, $B = 7$, $C = 1$, $D = 9$, and $E = 3$. With these values, we can see that each of the first 9 positive odd integers appears in the grid, and indeed, the sum of the numbers in each row, column, and main diagonal is the same.

Solution 2

In this solution we will determine the unknown values without finding that the row, column, and diagonal sum is 27. This solution will use more algebra.

Since the sum of the numbers in the third row is equal to the sum of the numbers in the third column, we know that

$$\begin{aligned} 11 + 13 + E &= B + 17 + E \\ 11 + 13 &= B + 17 \\ B &= 7 \end{aligned}$$

Again, since the sum of the numbers in the first row is equal to the sum of the numbers in the

first column, we know that

$$A + 5 + 7 = A + C + 11$$
$$5 + 7 = C + 11$$
$$C = 1$$

Here is the updated grid:

A	5	7
1	D	17
11	13	E

We also know that the two diagonals have the same sum, so we have

$$A + D + E = 7 + D + 11$$
$$A + E = 18$$

We have used the odd numbers 1, 5, 7, 11, 13, and 17. This leaves the odd numbers 3, 9 and 15 for A, D, and E. Since $A + E = 18$, A and E must be 3 and 15, in some order.

If $A = 3$ and $E = 15$, then the sum of the first row is $3 + 5 + 7 = 15$ and the sum of the third row is $11 + 13 + 15 = 39$. These sums are not the same. Therefore, $A = 3$ and $E = 15$ is not correct.

If $A = 15$ and $E = 3$, then the sum of the first row is $15 + 5 + 7 = 27$ and the sum of the third row is $11 + 13 + 3 = 27$. These sums are the same. Therefore, $A = 15$ and $E = 3$. This leaves $D = 9$.

Therefore, $A = 15$, $B = 7$, $C = 1$, $D = 9$, and $E = 3$. From here, one can easily show each row, column, and diagonal sums to 27, as we found in Solution 1.