Problem of the Week
Problem C and Solution
A Grand Sum

1000

Problem

Did you know that 1000 can be written as the sum of the 5 consecutive positive integers beginning with 198? That is,

$$1000 = 198 + 199 + 200 + 201 + 202$$

Also, 1000 can be written as the sum of 16 consecutive positive integers beginning with 55. That is,

$$1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70$$

It is also possible to write 1000 as a sum of 25 consecutive positive integers. This is the maximum number of consecutive positive integers that could be used to create the sum. Determine the smallest of the positive integers in this sum.

Solution

Solution 1

Let n, $n+1$, $n+2$, $\ldots$, $n+23$, and $n+24$ represent the 25 consecutive positive integers. Then,

$$\begin{aligned} n + n + 1 + n + 2 + \ldots + n + 23 + n + 24 &= 1000 \\ 25n + (1 + 2 + 3 + \ldots + 23 + 24) &= 1000 \\ 25n + 300 &= 1000 \\ 25n &= 700 \\ n &= 28 \end{aligned}$$

Therefore, the smallest integer in the sum is 28.

Note: A useful fact that we can use is that the sum of the first n natural numbers can be calculated using the formula $\frac{n(n+1)}{2}$.

Using the formula with $n = 24$, the sum $1 + 2 + 3 + \ldots + 23 + 24$ in the equation above can be quickly calculated as $\frac{24 \times 25}{2} = 300$.

Solution 2

Let n represent the middle integer of the 25 consecutive positive integers. Then there are 12 integers smaller than the middle integer, with the smallest integer being $(n-12)$, and 12 integers larger than the middle integer, with the largest integer being $(n+12)$.

Then, the sum of the 25 consecutive positive integers can be written as

$$(n-12)+(n-11)+\cdots+(n-1)+n+\ (n+1)+\cdots+(n+11)+(n+12)$$

This simplifies to $25n$, because for each positive integer 1 to 12 in the sum, the corresponding integer of opposite sign, -1 to -12, also appears.

Thus, we have

$$\begin{aligned} 25n &= 1000 \\ n &= 40 \\ n-12 &= 28 \end{aligned}$$

Therefore, the smallest integer in the sum is 28.

Solution 3

In this problem, we want to express 1000 as the sum of 25 consecutive positive integers. The average, $1000 \div 25 = 40$, is the middle integer in this sum. Solution 2 is a mathematical justification of this. There will be twelve consecutive integers above the average and twelve consecutive integers below the average. Therefore, the smallest integer in the sum is $40 - 12 = 28$.

Solution 4

Using the note that follows Solution 1, we know that the sum of the first 25 positive integers is

$$1+2+3+\cdots+24+25=\frac{25\times 26}{2}=325$$

Now, if we add 1 to each term in the sum, we get $2+3+4+\cdots+25+26=350$. Notice that the total increases by 25. In fact, every time we increase each term by 1, the total increases by 25.

The number of increases by 25 needed to get from 325 to 1000 is $\frac{1000-325}{25}=27$.

Therefore, we will need 27 increases for each term, and so the smallest number in the sum is $1+27=28$.