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Problem of the Week

Problem D and Solution

Hundred Deck 2

Problem

Hundred Deck is a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is red and the other side of the card is yellow.

Sarai places all of the cards on a table with each card's red side facing up. She first flips over every card that has a number on it which is a multiple of 2. She then flips over every card that has a number on it which is a multiple of 3. Finally, she flips over every card that has a number on it which is a multiple of 5.

After Sarai has finished, how many cards have their red side facing up?

Solution

After flipping over all of the cards with numbers that are multiples of 2, 50 cards have their red side facing up and 50 cards have their yellow side facing up. All of the cards with their red side facing up are numbered with an odd number. All of the cards with their yellow side facing up are numbered with an even number.

Next, in the second round of flips, Sarai flips over every card that is numbered with a multiple of 3. Let's look at how many cards with their red side facing up will be flipped over to yellow and how many cards with their yellow side facing up will be flipped over to red.

There are 33 multiples of 3 from 1 to 100. They are

$$3, 6, 9, 12, 15, \dots, 87, 90, 93, 96, 99$$

Of these numbers, 17 are odd and 16 are even. The 17 odd multiples of 3 currently have their red side facing up, and therefore are flipped over to yellow. The 16 even multiples of 3 currently have their yellow side facing up, and are therefore flipped over to red (again).

So, after the first flip there were 50 cards with their red side facing up and 50 cards with their yellow side facing up. Of the 50 red, 17 were flipped to yellow. Of the 50 yellow, 16 were flipped to red. Therefore, after the second round has finished, $50 - 17 + 16 = 49$ cards have their red side facing up and 51 cards have their yellow side facing up. The cards with their red side facing up are the cards with numbers that are odd and not a multiple of 3, or even and a multiple of 3. The cards with their yellow side facing up are the cards with numbers that are odd and a multiple of 3, or even and not a multiple of 3.



In the third round, Sarai flips all cards numbered with a multiple of 5. That is, she flips the cards numbered

5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100

For these 20 numbers, let's determine their colour after the last flip by considering four cases.

- Case 1: The number is odd and not a multiple of 3.
There are 7 cards that are numbered with a number that is a multiple of 5, odd, and not a multiple of 3. They are

5, 25, 35, 55, 65, 85, 95

Before the third flip, these 7 cards have their red side facing up, and are flipped to yellow in the third flip.

- Case 2: The number is even and a multiple of 3.
There are 3 cards that are numbered with a number that is a multiple of 5, even, and a multiple of 3. They are

30, 60, 90

Before the third flip, these 3 cards have their red side facing up, and are flipped to yellow in the third flip.

- Case 3: The number is odd and a multiple of 3.
There are 3 cards that are numbered with a number that is a multiple of 5, odd, and a multiple of 3. They are

15, 45, 75

Before the third flip, these 3 cards have their yellow side facing up, and are flipped to red in the third flip.

- Case 4: The number is even and not a multiple of 3.
There are 7 cards that are numbered with a number that is a multiple of 5, even, and not a multiple of 3. They are

10, 20, 40, 50, 70, 80, 100

Before the third flip, these 7 cards have their yellow side facing up, and are flipped to red in the third flip.

Therefore, after Sarai has finished, $49 - 7 - 3 + 3 + 7 = 49$ cards have their red side facing up.