Problem of the Week
Problem D and Solution
Five Digits

Problem
A sequence starts out with one 5, followed by two 6s, then three 7s, four 8s, five 9s, six 5s, seven 6s, eight 7s, nine 8s, ten 9s, eleven 5s, twelve 6s, and so on. (You should notice that only the five digits from 5 to 9 are used.)

The first 29 terms of the sequence appear below.

$$5, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 7, \ldots$$

Determine the 2022^{nd} digit in the sequence.

Solution
The first group in the sequence contains one 5. The second group in the sequence contains two 6s. To the end of the second group of digits, there is a total of $1 + 2 = 3$ digits. The third group in the sequence contains three 7s. To the end of the third group of digits, there is a total of $1 + 2 + 3 = 6$ digits. The n^{th} group in the sequence contains n digits. To the end of the n^{th} group of digits, there is a total of $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ digits.

How many groups of digits are required for there to be at least 2022 digits in the sequence?

We need to find the value of n so that $1 + 2 + 3 + \cdots + n \geq 2022$ and $1 + 2 + 3 + \cdots + (n - 1) < 2022$. At this point we will use trial and error. At the end of the solution, a more algebraic approach to finding the value of n using the quadratic formula is presented.

Suppose $n = 100$. Then $1 + 2 + 3 + \cdots + 100 = \frac{100(101)}{2} = 5050 > 2022$.

Suppose $n = 50$. Then $1 + 2 + 3 + \cdots + 50 = \frac{50(51)}{2} = 1275 < 2022$.

Suppose $n = 60$. Then $1 + 2 + 3 + \cdots + 60 = \frac{60(61)}{2} = 1830 < 2022$.

Suppose $n = 65$. Then $1 + 2 + 3 + \cdots + 65 = \frac{65(66)}{2} = 2145 > 2022$.

Suppose $n = 63$. Then $1 + 2 + 3 + \cdots + 63 = \frac{63(64)}{2} = 2016 < 2022$.

The 2022^{nd} digit is the sixth number in the next group of digits. That is, the 2022^{nd} digit is a digit in the 64^{th} group of digits.

Now, let's determine what digit is in the 64^{th} group of digits. Since we cycle through the digits and there are only five digits used, we can determine the digit by examining $\frac{64}{5} = 12\frac{4}{5}$. Thus, in the 64^{th} group of digits, the digit used is the 4^{th} digit in the sequence of digits. That is, in the 64^{th} group of digits, the digit used is an 8.

Since the 2022^{nd} digit is in the 64^{th} group of digits, it follows that the 2022^{nd} digit is an 8.

We will finish by showing how we can find the value of n algebraically.

We will first find the value of $n, n > 0$, so that

$$\begin{aligned}\frac{n(n+1)}{2} &= 2022\\ n(n+1) &= 4044\\ n^2 + n - 4044 &= 0\end{aligned}$$

The quadratic formula can be used to solve for n.

$$\begin{aligned}n &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-4044)}}{2}\\ &= \frac{-1 \pm \sqrt{16\,177}}{2}\end{aligned}$$

Since $n = \frac{-1-\sqrt{16\,177}}{2} < 0$, it is inadmissible.

Then $n = \frac{-1+\sqrt{16\,177}}{2} \approx 63.09$. But n is an integer. So, interpreting our result, when $n = 63$, the sum $1 + 2 + 3 + \cdots + 63 < 2022$, and when $n = 64$, the sum $1 + 2 + 3 + \cdots + 64 > 2022$. Thus, the 2022^{nd} digit is in the 64^{th} group of digits.