Problem of the Week
Problem D and Solution
The Dart Game

Problem

A carnival dart game has three non-overlapping circles in a rectangle. One circle has a value of 2, another has a value of 3, and the third has a value of 5. You are allowed to throw up to 10 darts, and you start the game with a running total of 0. If a dart lands in one of the circles, you add the value of the circle to the running total. If a dart does not land in one of the circles, then you do not add anything to the running total for that throw.

Suppose you have exactly 30 points after 10 throws. Let a represent the number of throws that landed in the circle with value 5, let b represent the number of throws that landed in the circle with value 3, and let c represent the number of throws that landed in the circle with value 2. Determine all possibilities for (a, b, c).

Solution

We need to determine all possibilities for (a, b, c) with $5a + 3b + 2c = 30$ and $a + b + c \leq 10$.

We will look at cases for a. Since $6 \times 5 = 30$, then the largest value of a is 6. The smallest value is $a = 0$ since $a \geq 0$.

Let's look at the specific case where $a = 2$ to develop a process for how to determine the number of ways to get a total of 30. We will use this process for all cases, but will not show our steps in the other cases.

If $a = 2$, this will account for a total of $2 \times 5 = 10$ points. Therefore, $30 - 10 = 20$ points will be needed from landing in the circles with values of 2 and 3.

Next, we find the maximum value for b, the number of throws that landed in the circle with value 3. We want b to account for a total that is less than or equal to 20, but also give a remainder that is even, since the remaining points need to come from the circle with value 2.

If $b = 7$, then this would give $7 \times 3 = 21$ points, which exceeds 20. When $b = 6$, then this would give $6 \times 3 = 18$ points. Then $c = 1$ would make the total exactly 30. Notice here that $a + b + c = 2 + 6 + 1 = 9 \leq 10$, as required. Thus, one possibility is that $a = 2$, $b = 6$, and $c = 1$.

We then need to replace circles with a value of 3 with circles with a value 2. We note that for every two circles with a value of 3, we have a total value of 6. We can replace those two circles with three circles of value 2. This means that $b = 6 - 2 = 4$ and $c = 1 + 3 = 4$. Notice here that $a + b + c = 2 + 4 + 4 = 10 \leq 10$, as required. Thus, another possibility is that $a = 2$, $b = 4$, and $c = 4$.

We can again replace two circles with a value of 3 with three circles of value 2. This means that $b = 4 - 2 = 2$ and $c = 4 + 3 = 7$. Notice here that $a + b + c = 2 + 2 + 7 = 11 > 10$. Therefore, this is not a possibility.

We can again replace two circles with a value of 3 with three circles of value 2. This means that $b = 2 - 2 = 0$ and $c = 7 + 3 = 10$. Notice here that $a + b + c = 2 + 0 + 10 = 12 > 10$. Therefore, this is not a possibility.

We cannot again replace two circles with a value of 3 with three circles of value 2 since this would make b negative.

To summarize, when $a = 2$, there are two combinations that give a total of 30 and have $a + b + c \leq 10$.

We use this process for all the possible values of a. Our results are summarized in the table below.

a	b	c	$5a + 3b + 2c$	$a + b + c$
6	0	0	30	6
5	1	1	30	7
4	2	2	30	8
4	0	5	30	9
3	5	0	30	8
3	3	3	30	9
3	1	6	30	10
2	6	1	30	9
2	4	4	30	10
1	7	2	30	10
0	10	0	30	10

We find that there are 11 possibilities for the number of throws that have landed in each circle. The 11 possibilities for (a, b, c) are:

$(6, 0, 0)$, $(5, 1, 1)$, $(4, 2, 2)$, $(4, 0, 5)$, $(3, 5, 0)$, $(3, 3, 3)$, $(3, 1, 6)$, $(2, 6, 1)$, $(2, 4, 4)$, $(1, 7, 2)$, $(0, 10, 0)$