Problem of the Week
Problem E and Solution
Three Squares

Problem

The three squares $ABCD$, $AEFG$, and $AHJK$ overlap as shown in the diagram.

The side length of each square, in centimetres, is a positive integer. The area of square $AEFG$ that is not covered by square $ABCD$ is $33$ cm$^2$. That is, the area of the shaded region $BEFGDC$ is $33$ cm$^2$. If $DG = GK$, determine all possible side lengths of each square.

Solution

Let $AD = x$ cm and $DG = y$ cm. Therefore $GK = DG = y$ cm.

Also, since the side length of each square is an integer, $x$ and $y$ are integers.

The shaded region has area $33$ cm$^2$. The shaded region is equal to the area of the square with side length $AG$ minus the area of the square with side length $AD$.

Since $AD = x$ and $AG = AD + DG = x + y$, we have

$$33 = (\text{area of square with side length } AG) - (\text{area of square with side length } AD)$$

$$= (x + y)^2 - x^2$$

$$= x^2 + 2xy + y^2 - x^2$$

$$= 2xy + y^2$$

$$= y(2x + y)$$

Since $x$ and $y$ are integers, so is $2x + y$. Therefore, $2x + y$ and $y$ are two positive integers that multiply to give $33$. Therefore, we must have $y = 1$ and $2x + y = 33$, or $y = 3$ and $2x + y = 11$, or $y = 11$ and $2x + y = 3$, or $y = 33$ and $2x + y = 1$. The last two would imply that $x < 0$, which is not possible. Therefore, $y = 1$ and $2x + y = 33$, or $y = 3$ and $2x + y = 11$.

When $y = 1$ and $2x + y = 33$, it follows that $x = 16$. Then square $ABCD$ has side length $x = 16$ cm, square $AEFG$ has side length $x + y = 17$ cm, and square $AHJK$ has side length $x + 2y = 18$ cm.

When $y = 3$ and $2x + y = 11$, it follows that $x = 4$. Then square $ABCD$ has side length $x = 4$ cm, square $AEFG$ has side length $x + y = 7$ cm, and square $AHJK$ has side length $x + 2y = 10$ cm.

Therefore, there are two possible sets of squares. The squares are either $16$ cm $\times$ $16$ cm and $17$ cm $\times$ $17$ cm and $18$ cm $\times$ $18$ cm, or $4$ cm $\times$ $4$ cm and $7$ cm $\times$ $7$ cm and $10$ cm $\times$ $10$ cm. Each of these sets of squares satisfies the conditions of the problem.