Problem of the Week
3, 6, 9, \ldots, 2400

Problem E and Solution
All Square

Problem
The positive multiples of 3 from 3 to 2400, inclusive, are each multiplied by the same positive integer, $n$. All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer $n$ that makes this true.

Solution
What does the prime factorization of a perfect square look like? Let’s look at a few examples: $9 = 3^2$, $36 = 6^2 = 2^23^2$, and $129\,600 = 360^2 = 2^63^45^2$. Notice that the exponent on each of the prime factors in the prime factorization in each of the three examples is an even number. In fact, a positive integer is a perfect square exactly when the exponent on each prime in its prime factorization is even. Can you convince yourself that this is true?

The positive integer $n$ is the smallest positive integer such that

$$3n + 6n + 9n + \cdots + 2394n + 2397n + 2400n$$

is a perfect square.

Factoring expression (1), we obtain

$$3n(1 + 2 + 3 + \cdots + 798 + 799 + 800)$$

Then, using the formula $1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$, with $n = 800$, we see that this expression is equal to

$$3n \left(\frac{800 \times 801}{2}\right) = 3n(400)(801)$$

Factoring $3 \times 400 \times 801$ into the product of primes, we have that expression (1) is equal to


We need to determine what additional factors are required to make the quantity in expression (2) a perfect square such that $n$ is as small as possible. In order for the exponent on each prime in the prime factorization to be even, we need $n$ to be $3 \times 89 = 267$. Then the quantity in expression (2) is the perfect square


Therefore, the smallest positive integer is 267 and the perfect square is

$$267 \times 3 \times 400 \times 801 = 256\,640\,400 = (16\,020)^2$$
NOTE: When solving this problem, we could have instead noticed that
$3n + 6n + 9n + \cdots + 2400n$ is an arithmetic series with $t_1 = 3n$ and $t_{800} = 2400n$.
Substituting these values for $t_1$ and $t_{800}$ into the formula for the sum of the terms in an
arithmetic series, we get

$$S_{800} = \frac{800}{2} (3n + 2400n) = 400(2403n)$$

When we factor $400(2403n)$ into the product of primes, we get the same expression as (2), and
then we can continue from there to get the solution of 267.