Problem of the Week
Problem D and Solution
They’re Blue

Problem
In rectangle $ABCD$, the length of side $AB$ is 7 m and the length of side $BC$ is 5 m. Four points, $W$, $X$, $Y$, and $Z$, lie on diagonal $BD$, dividing it into five equal segments. Triangles $AWX$, $AYZ$, $CWX$, and $CYZ$ are then painted blue, as shown. Determine the area of the painted region.

Solution
Solution 1
Using the formula for area of a triangle, $\text{area} = \frac{\text{base} \times \text{height}}{2}$, we have $\triangle ABD = \frac{7 \times 5}{2} = \frac{35}{2} \text{ m}^2$.
The five triangles $\triangle ADW$, $\triangle AWX$, $\triangle AXY$, $\triangle AYZ$, and $\triangle ABZ$ have the same height, which is equal to the perpendicular distance between $BD$ and $A$. Since $DW = WX = XY = YZ = ZB$, it follows that the five triangles also have equal bases. Therefore, the area of each of these five triangles is equal to $\frac{1}{5}(\text{area } \triangle ABD) = \frac{1}{5} \left( \frac{35}{2} \right) = \frac{7}{2} \text{ m}^2$.

Similarly, the area of $\triangle BCD$ is equal to $\frac{7 \times 5}{2} = \frac{35}{2} \text{ m}^2$. The five triangles $\triangle CDW$, $\triangle CWX$, $\triangle CXY$, $\triangle CYZ$, and $\triangle CBZ$ also have the same height and equal bases. Therefore, the area of each of these five triangles is equal to $\frac{1}{5}(\text{area } \triangle BCD) = \frac{1}{5} \left( \frac{35}{2} \right) = \frac{7}{2} \text{ m}^2$.

Therefore, the area of the painted region is $4 \left( \frac{7}{2} \right) = 14 \text{ m}^2$.

Solution 2
Since $ABCD$ is a rectangle, $\angle DAB = 90^\circ$, so $\triangle ABD$ is a right-angled triangle. We can then use the Pythagorean Theorem to calculate $BD^2 = AB^2 + AD^2 = 7^2 + 5^2 = 49 + 25 = 74$, and so $BD = \sqrt{74}$, since $BD > 0$. Therefore, $DW = WX = XY = YZ = ZB = \frac{1}{5}(BD) = \frac{1}{5}\sqrt{74}$.

Using the formula for area of a triangle, $\text{area} = \frac{\text{base} \times \text{height}}{2}$, we have $\triangle ABD = \frac{7 \times 5}{2} = \frac{35}{2} \text{ m}^2$.

Let’s treat $BD = \sqrt{74}$ as the base of $\triangle ABD$ and let $h$ be the corresponding height. Since the area of $\triangle ABD$ is $\frac{35}{2}$, then we have $\frac{\sqrt{74} \times h}{2} = \frac{35}{2}$ and so $\sqrt{74} \times h = 35$, thus $h = \frac{35}{\sqrt{74}}$.

$\triangle AWX$ and $\triangle AYZ$ both have height $h = \frac{35}{\sqrt{74}}$ and base $\frac{\sqrt{74}}{5}$, so
$\text{area } \triangle AWX = \text{area } \triangle AYZ = \frac{1}{2} \left( \frac{\sqrt{74}}{5} \right) \left( \frac{35}{\sqrt{74}} \right) = \frac{7}{2} \text{ m}^2$.

Similarly, $\triangle CWX$ and $\triangle CYZ$ both have height $h = \frac{35}{\sqrt{74}}$ and base $\frac{\sqrt{74}}{5}$, so
$\text{area } \triangle CWX = \text{area } \triangle CYZ = \frac{1}{2} \left( \frac{\sqrt{74}}{5} \right) \left( \frac{35}{\sqrt{74}} \right) = \frac{7}{2} \text{ m}^2$.

Therefore, the area of the painted region is $4 \left( \frac{7}{2} \right) = 14 \text{ m}^2$. 